Fundamentals of Optical Sciences

WS 2017/2018 **4. Exercise sheet** **08.11.2017**

Lecture: Prof. Dr. Alejandro Saenz, Dr. Sven Ramelow

Deliver your answers on 15.11.2017.

Problem 1: Plane Waves

Consider a plane wave in vacuum in complex notation. The magnetic field is given by

$$\mathbf{B}(x,z,t) = (0, A\mathrm{e}^{i\alpha x + i\beta z}, 0)\ \mathrm{e}^{-i\omega t} = (0, B_y(x,z,t), 0)$$

where $A, \alpha, \beta \in \mathbb{R}$ and $k^2 = \alpha^2 + \beta^2$.

a) Show that $\mathbf{B}(x,z,t)$ fulfills the wave equation. What conditions need to be fulfilled by α, β, and ω?

b) $\mathbf{B}(x,z,t)$ constitutes together with the electric field

$$\mathbf{E}(x,z,t) = \frac{1}{\omega\varepsilon_0\mu_0}(\beta B_y(x,z,t), 0, -\alpha B_y(x,z,t))$$

a transversal plane wave. Calculate the time-averaged Poynting vector $\langle \mathbf{S}(x,z,t)\rangle_t$ for the given fields $\mathbf{B}(x,z,t)$ and $\mathbf{E}(x,z,t)$.

Problem 2: Fourier transform

a) Consider the Fourier transforms $\tilde{f}_1(k)$ and $\tilde{f}_2(k)$ of two functions $f_1(x)$ and $f_2(x)$, respectively:

$$\tilde{f}_{1,2}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x\ \mathrm{e}^{-\mathrm{i}kx} f_{1,2}(x).$$

What is the Fourier transform $\tilde{g}(k)$ of the product of these functions

$$g(x) = f_1(x) f_2(x) \quad ?$$

b) What is the Fourier transform of

$$f(x) = \mathrm{e}^{-|x|}$$

(... continuation on the next page ...)

c) Prove that for every square integrable function $f(x)$ the Parseval's theorem

$$\int_{-\infty}^{\infty} \mathrm{d}x \ |f(x)|^2 = \int_{-\infty}^{\infty} \mathrm{d}k \ \left|\tilde{f}(k)\right|^2$$

holds, where $\tilde{f}(k)$ is the Fourier transform of $f(x)$.

d) If $f(x)$ is a real function what condition has to be fulfilled by its Fourier transform $\tilde{f}(k)$?

Problem 3: Fourier Series and Gibbs Phenomenon

a) Show that the Fourier series of the periodic function $f(x) = f(x+2\pi)$:

$$f(x) = \begin{cases} -1 & -\pi \leq x < 0 \\ 1 & 0 \leq x < \pi \end{cases} \tag{1}$$

can be expressed by

$$f(x) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin[(2n+1)x]}{2n+1}, \quad n \in \mathbb{Z}. \tag{2}$$

b) The Fourier series of (2) can be reformulated as the limit of a partial sum $s_k(x)$

$$f(x) = \lim_{k\to\infty} s_k(x)$$

where

$$s_k(x) = \frac{4}{\pi} \sum_{n=0}^{k} \frac{\sin[(2n+1)x]}{2n+1}. \tag{3}$$

The partial sum $s_k(x)$ has an extremum at every $x_k = \frac{\pi}{2k}, \forall k$. Evaluate the partial sum (3) at these positions. Under which condition can the Fourier series be expressed by the integral

$$I = \frac{2}{\pi} \int_0^{\pi} \mathrm{d}t \ \frac{\sin(t)}{t} \quad ? \tag{4}$$

c) The integral (4) equals a constant value of $C \approx 1.17898$. What does this mean for the approximation of the function (1) by Fourier polynomials with respect to the approximation order?