Problem 1: Gaussian Beams 1

Consider the complex amplitude of a Gaussian beam:

\[ U(r) = \frac{W_0}{W(z)} \exp \left[ -\frac{r^2}{W(z)^2} \right] \exp \left[ (-\imath k z - \imath k \frac{r^2}{2 R(z)} + \imath \tan^{-1} \left( \frac{z}{z_0} \right)) \right] = A(r) e^{-\imath k z}. \]

With the beam waist radius \( W_0 \), the Rayleigh length \( z_0 \), the beam waist along the optical axis \( W(z) \) and the wave front radius \( R(z) \) defined as:

\[
\begin{align*}
W_0 &= \sqrt{\frac{\lambda z_0}{\pi}} \\
z_0 &= \frac{\pi W_0^2}{\lambda} \\
W(z) &= W_0 \sqrt{1 + \left( \frac{z}{z_0} \right)^2} \\
R(z) &= z \left[ 1 + \left( \frac{z_0}{z} \right)^2 \right]
\end{align*}
\]

Show that the complex envelope \( A(r) \) of this beam satisfies the paraxial Helmholtz equation:

\[
(\nabla_T^2 - 2ik \frac{\partial}{\partial z})A(r) = 0
\]

with \( \nabla_T^2 \) being the transverse part of the Laplacian operator.

Hint 1: You might show first, that \( A(r) \) can be expressed as \( A(r) = \frac{q(0)}{q(z)} e^{-\imath k z_0^2 / z} \) with \( q(z) = z + \imath z_0 \).

Hint 2: Use an appropriate coordinate system.
Problem 2: Gaussian Beams 2

Consider the same Gauss beam as in problem 1.

a) What is the meaning of $z_0$ for the beam waist $W(z)$?

b) Show that in the far-field, the beam propagates in the form of a cone of half angle \[ \tan \Theta_0 = \frac{\lambda}{\pi W_0} \]

c) Discuss the radius of curvature $R(z)$ of the wavefronts in the near- and far-field. At which length does maximum wave-front curvature occur?

Problem 3: Gaussian Beams 3

The Gaussian beam described above represents the fundamental mode. Higher order modes, so-called Hermite-Gaussian modes, are expressed as:

\[
E_{l,m}(x, y, z) = E_0 \frac{W_0}{W(z)} H_l\left(\frac{\sqrt{2} x}{W(z)}\right) H_m\left(\frac{\sqrt{2} y}{W(z)}\right) \exp\left(-\frac{r^2}{W^2(z)}\right) \exp\left(-i \frac{k r^2}{2 R(z)}\right) \exp\left(-ikz + i(1 + l + m) \tan^{-1}\left(\frac{z}{z_0}\right)\right)
\]

where the modes are indexed TEM$_{l,m}$. Plot the intensity of the following modes: TEM$_{0,0}$, TEM$_{1,0}$, TEM$_{0,1}$, and TEM$_{1,1}$. Use a computer.