
Fundamentals of Optical Sciences

WS 2017/2018

2. Exercise sheet

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Deliver your answers on 01.11.2017.

Problem 1: Wave Packet

The one-dimensional wave packet

$$p(z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk A(k) e^{i(kz - \omega(k)t)} \quad (1)$$

in a dispersive medium with

$$\omega(k) = \alpha k^2, \quad \alpha = \text{const.} \quad (2)$$

possesses at $t = 0$ the shape of a Gaussian,

$$p(z, 0) = C e^{-\frac{z^2}{2\Delta^2}} e^{ik_0 z}, \quad (3)$$

with the constants C , Δ , and k_0 .

a) Show that the weight function

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dz p(z, 0) e^{-ikz} \quad (4)$$

is also a Gaussian. What is the width Δk of $|A(k)|^2$? Let the width be defined as the distance Δk of the k values at which the function value has dropped to the fraction $1/e$ of its maximum.

Hint:
$$\int_{-\infty}^{+\infty} e^{-(ax^2+bx)} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$$

b) Determine the full spatial and temporal dependence of $p(z, t)$.

c) Calculate the time dependence of the width $\Delta z(t)$ (defined analogous to Δk) of $|p(z, t)|^2$. What is the velocity with which the maximum of $|p(z, t)|^2$ moves? Compare it to the group velocity.

Problem 2: Coulomb and Lorenz Gauge

A vector potential $\mathbf{A}(\mathbf{r}, t)$ and a scalar potential $\phi(\mathbf{r}, t)$ are given which should be transformed into a different gauge using the scalar field $\chi(\mathbf{r}, t)$.

- a) Which equation has to be satisfied by χ if the gauged potentials satisfy the Lorenz condition? Why is the Lorenz gauge denoted as a *gauge class* ?
- b) Which equation has to be satisfied by χ to transform into the Coulomb gauge? Is this equation always solvable?

Problem 3: Magnetic Monopoles

Consider the following Maxwell equations with the electric field \mathbf{E} , the magnetic induction \mathbf{B} , the dielectric displacement \mathbf{D} , and the magnetic field strength \mathbf{H} ,

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_e & \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_e \\ \nabla \cdot \mathbf{B} &= \rho_m & -\nabla \times \mathbf{E} &= \frac{\partial \mathbf{B}}{\partial t} + \mathbf{J}_m, \end{aligned}$$

which allow for the existence of magnetic monopoles with the magnetic charge density $\rho_m(\mathbf{r})$ and the magnetic current density $\mathbf{J}_m(\mathbf{r})$.

- a) Show that the Maxwell equations are form-invariant under the transformation

$$\begin{aligned} \mathbf{E} &= \mathbf{E}' \cos \xi + Z \mathbf{H}' \sin \xi & \mathbf{D} &= \mathbf{D}' \cos \xi + Z^{-1} \mathbf{B}' \sin \xi \\ \mathbf{H} &= -Z^{-1} \mathbf{E}' \sin \xi + \mathbf{H}' \cos \xi & \mathbf{B} &= -Z \mathbf{D}' \sin \xi + \mathbf{B}' \cos \xi \end{aligned}$$

(where $Z = \sqrt{\mu_0/\epsilon_0}$) under the condition that the charge density and the current density transform as

$$\begin{aligned} \rho_e &= \rho'_e \cos \xi + Z^{-1} \rho'_m \sin \xi & \mathbf{J}_e &= \mathbf{J}'_e \cos \xi + Z^{-1} \mathbf{J}'_m \sin \xi \\ \rho_m &= -Z \rho'_e \sin \xi + \rho'_m \cos \xi & \mathbf{J}_m &= -Z \mathbf{J}'_e \sin \xi + \mathbf{J}'_m \cos \xi. \end{aligned}$$

- b) The Lorentz force acting on a particle with electric charge q_e and magnetic charge q_m is given by

$$\mathbf{F} = q_e (\mathbf{E} + \mathbf{v} \times \mathbf{B}) + q_m (\mathbf{H} - \mathbf{v} \times \mathbf{D}).$$

Show that the Lorentz force is invariant under the transformations above.

- c) What are the consequences of the invariance of the transformations for the existence of magnetic monopoles?