Exercise 1: Coulomb and Lorenz Gauges

A vector potential $\mathbf{A}(\mathbf{r}, t)$ and a scalar potential $\phi(\mathbf{r}, t)$ are given which should be transformed into a different gauge using the scalar field $\chi(\mathbf{r}, t)$.

a) Which equation has to be satisfied by $\chi$ if the gauged potentials satisfy the Lorenz condition? Why is the Lorenz gauge denoted as a "gauge class"?

b) Which equation has to be satisfied by $\chi$ to transform into the Coulomb gauge? Is this equation always solvable?
Exercise 2: Magnetic Monopoles

Consider the following Maxwell equations with the electric field $E$, the magnetic induction $B$, the dielectric displacement $D$, and the magnetic field strength $H$,

$$\nabla \cdot D = \rho_e \quad \nabla \times H = \frac{\partial D}{\partial t} + J_e$$

$$\nabla \cdot B = \rho_m \quad -\nabla \times E = \frac{\partial B}{\partial t} + J_m,$$

which allow for the existence of magnetic monopoles with the magnetic charge density $\rho_m(r)$ and the magnetic current density $J_m(r)$.

a) Show that the Maxwell equations are form-invariant under the transformation

$$E = E' \cos \xi + Z H' \sin \xi \quad D = D' \cos \xi + Z^{-1} B' \sin \xi$$

$$H = -Z^{-1} E' \sin \xi + H' \cos \xi \quad B = -Z D' \sin \xi + B' \cos \xi,$$

(where $Z = \sqrt{\mu_0/\epsilon_0}$) under the condition that the charge density and the current density transform as

$$\rho_e = \rho_e' \cos \xi + Z^{-1} \rho_m' \sin \xi \quad J_e = J_e' \cos \xi + Z^{-1} J_m' \sin \xi$$

$$\rho_m = -Z \rho_e' \sin \xi + \rho_m' \cos \xi \quad J_m = -Z J_e' \sin \xi + J_m' \cos \xi.$$

b) The Lorentz force acting on a particle with electric charge $q_e$ and magnetic charge $q_m$ is given by

$$F = q_e (E + v \times B) + q_m (H - v \times D).$$

Show that the Lorentz force is invariant under the transformations above.

c) What are the consequences of the invariance of the transformations for the existence of magnetic monopoles?
Exercise 3: Wave Packet

The one-dimensional wave packet

\[ p(z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \, A(k) \, e^{i(kz - \omega(k)t)} \]  \hspace{1cm} (1)

in a dispersive medium with

\[ \omega(k) = \alpha k^2, \quad \alpha = \text{const.} \]  \hspace{1cm} (2)

possesses at \( t = 0 \) the shape of a Gaussian,

\[ p(z, 0) = C \, e^{-\frac{z^2}{2\Delta^2}} \, e^{i k_0 z} \]  \hspace{1cm} (3)

with the constants \( C, \Delta, \) and \( k_0 \).

a) Show that the weight function

\[ A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dz \, p(z, 0) \, e^{-ikz} \]  \hspace{1cm} (4)

is also a Gaussian. What is the width \( \Delta k \) of \( |A(k)|^2 \)? Let the width be defined as the distance \( \Delta k \) of the \( k \) values at which the function value has dropped to the fraction \( 1/e \) of its maximum.

*Hint: \( \int_{-\infty}^{+\infty} e^{-ax^2 + bx} \, dx = \sqrt{\frac{\pi}{a}} \, e^{\frac{b^2}{4a}} \)*

b) Determine the full spatial and temporal dependence of \( p(z, t) \).

c) Calculate the time dependence of the width \( \Delta z(t) \) (defined analogous to \( \Delta k \)) of \( |p(z, t)|^2 \). What is the velocity with which the maximum of \( |p(z, t)|^2 \) moves? Compare it to the group velocity.