Fundamentals of Optical Sciences

WS 2019/2020 **1 . Exercise** **18.10.2019**

Lecture: Prof. Dr. Alejandro Saenz, Prof. Dr. Oliver Benson

Hand in your answers before the lecture on 25.10.2019.

Exercise 1: Coulomb and Lorenz Gauges

A vector potential $\mathbf{A}(\mathbf{r}, t)$ and a scalar potential $\phi(\mathbf{r}, t)$ are given which should be transformed into a different gauge using the scalar field $\chi(\mathbf{r}, t)$.

a) Which equation has to be satisfied by χ if the gauged potentials satisfy the Lorenz condition? Why is the Lorenz gauge denoted as a *gauge class* ?

b) Which equation has to be satisfied by χ to transform into the Coulomb gauge? Is this equation always solvable?

Exercise 2: Magnetic Monopoles

Consider the following Maxwell equations with the electric field $\mathbf{E}$, the magnetic induction $\mathbf{B}$, the dielectric displacement $\mathbf{D}$, and the magnetic field strength $\mathbf{H}$,

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_e & \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_e \\ \nabla \cdot \mathbf{B} &= \rho_m & -\nabla \times \mathbf{E} &= \frac{\partial \mathbf{B}}{\partial t} + \mathbf{J}_m\,, \end{aligned}$$

which allow for the existence of magnetic monopoles with the magnetic charge density $\rho_m(\mathbf{r})$ and the magnetic current density $\mathbf{J_m}(\mathbf{r})$.

a) Show that the Maxwell equations are form-invariant under the transformation

$$\begin{aligned} \mathbf{E} &= \mathbf{E}'\cos\xi + Z\mathbf{H}'\sin\xi & \mathbf{D} &= \mathbf{D}'\cos\xi + Z^{-1}\mathbf{B}'\sin\xi \\ \mathbf{H} &= -Z^{-1}\mathbf{E}'\sin\xi + \mathbf{H}'\cos\xi & \mathbf{B} &= -Z\mathbf{D}'\sin\xi + \mathbf{B}'\cos\xi \end{aligned}$$

(where $Z = \sqrt{\mu_0/\epsilon_0}$) under the condition that the charge density and the current density transform as

$$\begin{aligned} \rho_e &= \rho'_e\cos\xi + Z^{-1}\rho'_m\sin\xi & \mathbf{J}_e &= \mathbf{J}'_e\cos\xi + Z^{-1}\mathbf{J}'_m\sin\xi \\ \rho_m &= -Z\rho'_e\sin\xi + \rho'_m\cos\xi & \mathbf{J}_m &= -Z\mathbf{J}'_e\sin\xi + \mathbf{J}'_m\cos\xi\,. \end{aligned}$$

b) The Lorentz force acting on a particle with electric charge q_e and magnetic charge q_m is given by

$$\mathbf{F} = q_e\,(\mathbf{E} + \mathbf{v}\times\mathbf{B}) + q_m\,(\mathbf{H} - \mathbf{v}\times\mathbf{D})\,.$$

Show that the Lorentz force is invariant under the transformations above.

c) What are the consequences of the invariance of the transformations for the existence of magnetic monopoles?

Exercise 3: Wave Packet

The one-dimensional wave packet

$$p(z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}k\; A(k)\, e^{i(kz-\omega(k)t)} \tag{1}$$

in a dispersive medium with

$$\omega(k) = \alpha k^2, \qquad \alpha = \text{const.} \tag{2}$$

possesses at $t = 0$ the shape of a Gaussian,

$$p(z,0) = C\, e^{-\frac{z^2}{2\Delta^2}}\, e^{ik_0 z}\,, \tag{3}$$

with the constants C, Δ, and k_0.

a) Show that the weight function

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}z\; p(z,0)\, e^{-ikz} \tag{4}$$

is also a Gaussian. What is the width Δk of $|A(k)|^2$? Let the width be defined as the distance Δk of the k values at which the function value has dropped to the fraction $1/e$ of its maximum.

Hint: $\displaystyle\int_{-\infty}^{+\infty} e^{-(ax^2+bx)}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}\; e^{\frac{b^2}{4a}}$

b) Determine the full spatial and temporal dependence of $p(z,t)$.

c) Calculate the time dependence of the width $\Delta z(t)$ (defined analogous to Δk) of $|p(z,t)|^2$. What is the velocity with which the maximum of $|p(z,t)|^2$ moves? Compare it to the group velocity.