Quantum Information und Quantum Computers

SS 2018 **2 . Exercise** **26.04.2018**

Task 1

The experimental confirmation of the non-locality of quantum mechanics was first achieved using *Bell's inequalities*. A disadvantage of such experiments is the fact that a single measurement does not provide an answer, since the proof of non-locality is based on inequalities for expectation values. Therefore, a series of measurements is required that shows the violation or fulfilment of the inequalities based on an error analysis and thus within some standard deviation. An alternative was shown by Mermin ("Bell's inequalities without inequalities"). In this case one considers a system of, e. g., three spins (A, B, and C) that are prepared in the GHZ state (Greenberger, Horne, Zeilinger)

$$|\,\psi\,\rangle \;=\; \frac{1}{\sqrt{2}}\,(|\,000\,\rangle - |\,111\,\rangle) \qquad .$$

a) Write $|\,\psi\,\rangle$ in vector form.

b) Write the operators

(i) $\hat{\mathrm{P}} = \hat{\sigma}_x \otimes \hat{\sigma}_y \otimes \hat{\sigma}_y$

(ii) $\hat{\mathrm{Q}} = \hat{\sigma}_y \otimes \hat{\sigma}_x \otimes \hat{\sigma}_y$

(iii) $\hat{\mathrm{R}} = \hat{\sigma}_y \otimes \hat{\sigma}_y \otimes \hat{\sigma}_x$

(iv) $\hat{\mathrm{S}} = \hat{\sigma}_x \otimes \hat{\sigma}_x \otimes \hat{\sigma}_x$

in matrix form.

The Pauli matrices are:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad .$$

c) Apply each of the operators $\hat{\mathrm{P}}, \hat{\mathrm{Q}}, \hat{\mathrm{R}}$, and $\hat{\mathrm{S}}$ on $|\,\psi\,\rangle$.

(continued on the next page . . .)

d) At the heart of the Einstein-Bohr debate was the question whether the result of a quantum measurement is a consequence of the measurement process itself (Bohr) or whether the result is independent of the measurement (Einstein). Einstein called this independence "reality" in the sense that the state of any system is well defined at all times and thus in an ideal measurement a deterministic result is obtained. In the case of the GHZ states this "reality" postulate of Einstein implies that the results of ideal measurements with respect to the three Pauli matrices $\hat{\sigma}_i$ contained in any of the four operators $\hat{\mathrm{P}}, \hat{\mathrm{Q}}, \hat{\mathrm{R}}$, and $\hat{\mathrm{S}}$ are predetermined and thus fixed to the values they had before the measurements were performed. In other words, the expectation value of a measurement of, e. g., spin A with respect to $\hat{\sigma}_x$ for some state $|\,\phi\,\rangle$ is fixed to the value m_x^A, independently whether the state of the three-spin system is measured (or not measured) with respect to $\hat{\mathrm{P}}, \hat{\mathrm{Q}}, \hat{\mathrm{R}}$, or $\hat{\mathrm{S}}$.

Show that the assumption of "reality" of the measurement outcome with respect to the three Pauli matrices leads for a system in the GHZ state to a contradiction to the predictions of quantum mechanics. For this purpose, multiply the expectation values m_j^N for the Pauli matrices contained in $\hat{\mathrm{P}}$ and $\hat{\mathrm{Q}}$ with each other and compare it with the corresponding result for the Pauli matrices contained in $\hat{\mathrm{R}}$ and $\hat{\mathrm{S}}$.

Task 2

In the theory of universal computers the reversible Fredkin and Toffoli gates play an important role. Show how the Fredkin gate can be used to emulate

a) an *AND* or

b) a *SWAP* (*CROSSOVER*) gate in which the values of two bits are exchanged.

c) In which way the Toffoli gate can be used to generate a copy function (*COPY* / *FANOUT*)?

Task 3

Give the matrix representation for the following qubit gates (verify your results!):

a) Fredkin gate,

b) Toffoli gate,

c) *SWAP* gate (exchange of the two input qubits) and

d) C-Hadamard ("controlled Hadamard") gate, i. e. one qubit serves as control qubit: only if it is in state $|\,1\,\rangle$ the Hadamard operation should be applied to the second (the target) qubit. The Hadamard gate generates the state $\psi = (|\,0\,\rangle + |\,1\,\rangle)/\sqrt{2}$ for the input state $|\,0\,\rangle$ and $\psi = (|\,0\,\rangle - |\,1\,\rangle)/\sqrt{2}$ for the input state $|\,1\,\rangle$.