Quantum Information and Quantum Computer

SS 2018 **8 . Exercise** **14.06.2018**

Task 1

For the construction of complicated algorithms not only the 1-qubit gates but also the conditional 2-qubit gates play an important role. However, the often adopted universal quantum gate contains only a single conditional 2-qubit gate, the $CNOT$. How is it possible to formulate arbitrary conditional 2-qubit gates, i. e. 2-qubit gates with a control and a target qubit in which a specific unitary transformation is applied to the target qubit, if the control qubit is in the state $|\,1\,\rangle$? A derivation is based on the following two propositions:

Proposition 1: Let $\hat{\mathrm{U}}$ describe a unitary 1-qubit transformation. Then there is a set of real numbers α, β, γ, and δ which allow for writing any $\hat{\mathrm{U}}$ as

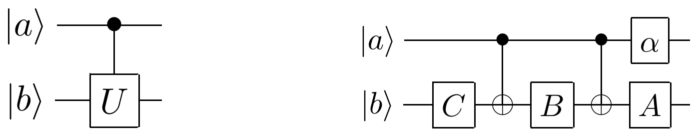
$$\hat{\mathrm{U}} \;=\; e^{i\alpha}\,\hat{\mathrm{R}}_z(\beta)\,\hat{\mathrm{R}}_y(\gamma)\,\hat{\mathrm{R}}_z(\delta)$$

with the rotations $\hat{\mathrm{R}}_x(\alpha)$ by the angle α around the x axis.

Proposition 2: Let $\hat{\mathrm{U}}$ describe a unitary 1-qubit transformation. Then there is a set of unitary 1-qubit operators $\hat{\mathrm{A}}, \hat{\mathrm{B}}$, and $\hat{\mathrm{C}}$ for which $\hat{\mathrm{A}}\hat{\mathrm{B}}\hat{\mathrm{C}} = \hat{\mathrm{I}}$ and $\hat{\mathrm{U}} = e^{i\alpha}\,\hat{\mathrm{A}}\hat{\mathrm{X}}\hat{\mathrm{B}}\hat{\mathrm{X}}\hat{\mathrm{C}}$ with the phase factor α is valid.

a) Proof proposition 2 by using proposition 1 with $\hat{\mathrm{A}} = \hat{\mathrm{R}}_z(\beta)\,\hat{\mathrm{R}}_y(\gamma/2)$, $\hat{\mathrm{B}} = \hat{\mathrm{R}}_y(-\gamma/2)\,\hat{\mathrm{R}}_z(-(\delta+\beta)/2)$ and $\hat{\mathrm{C}} = \hat{\mathrm{R}}_z((\delta-\beta)/2)$.

b) Explain the equivalence of the two circuits



where α stands for the qubit operation $\hat{\alpha} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$.

Task 2

a) Verify the following useful identities:
$\hat{\mathrm{H}}\hat{\mathrm{X}}\hat{\mathrm{H}} = \hat{\mathrm{Z}}$, $\hat{\mathrm{H}}\hat{\mathrm{Y}}\hat{\mathrm{H}} = -\hat{\mathrm{Y}}$, and $\hat{\mathrm{H}}\hat{\mathrm{Z}}\hat{\mathrm{H}} = \hat{\mathrm{X}}$.

b) Use the result obtained in a) to show that, besides a possible global phase factor, the relation $\hat{\mathrm{H}}\hat{\mathrm{T}}\hat{\mathrm{H}} = \hat{\mathrm{R}}_x(\pi/4)$ holds.

c) In order to achieve a universal quantum gate with the set of gates $\{\hat{\mathrm{H}}, \hat{\mathrm{S}}, \hat{\mathrm{T}}, \hat{\mathrm{U}}_{\mathrm{CNOT}}\}$ it must be possible to perform also a qubit rotation by an arbitrary angle. To achieve this goal, it must be possible to perform a rotation by an angle α that is not commensurable with 2π, i. e. α is an irrational fraction of 2π. Show that such a rotation is achieved by the transformationen $\hat{\mathrm{U}} = \hat{\mathrm{T}}\hat{\mathrm{H}}\hat{\mathrm{T}}\hat{\mathrm{H}}$. Asked differently, which rotation is associated with the transformation $\hat{\mathrm{U}}$?