# Basics of Lattice Quantum Field Theory*

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**Abstract**

3+1 lectures for the Graduiertenkolleg 'Mass, Spectrum, Symmetry', in October 2010 are based on these notes.

## 0 General remarks

Some abbreviations:

PT = perturbative, perturbation theory
NP = non PT
QFT = quantum field theory
LQFT = Lattice QFT
QM = Quantum Mechanics
UV, IR = ultraviolet, infrared
units: $\hbar = 1, \quad c = 1, \quad$ energy $\sim$ mass $\sim$ momentum $\sim$ length$^{-1}$

- general principles QFT $\leftrightarrow$ LQFT explained for $\varphi^4$
- later: discretization of gluons and quarks for QCD

$\Rightarrow$ vertical spaces in the text below are for your hand-written decorations

*. This document has been written using the GNU TeXmacs text editor (see `www.texmacs.org`).

# 1 Lecture: Summary PT QFT & why NP

## 1.1 Outline PT QFT

we emphasize:

- PT QFT (incl. renormalization) = algorithm to generate the PT series in the *renormalized* coupling
- start from Euclidean ($\rightarrow$Wick rotation) action:

$$S[\varphi]=\int d^4x\left\{\frac{1}{2}(\partial_\mu\varphi)^2+\frac{m_0^2}{2}\varphi^2+\frac{g_0}{4!}\varphi^4\right\}$$

- O(4) invariant ($\rightarrow$Lorentz)
- $\infty$ volume
- internal Z(2), phases: massive unbroken, massless, massive broken
- physics $\leftrightarrow$correlations

$$\langle\varphi(x^{(1)})\varphi(x^{(2)})\cdots\varphi(x^{(n)})\rangle=\frac{1}{Z}\int D\varphi\,\mathrm{e}^{-S[\varphi]}\varphi(x^{(1)})\varphi(x^{(2)})\cdots\varphi(x^{(n)})$$

summarized in generating functional

$$\mathrm{e}^{W[j]}=\left\langle\mathrm{e}^{\int d^4x\,j(x)\varphi(x)}\right\rangle,\quad W[0]=0,$$

$$\langle\varphi(x^{(1)})\varphi(x^{(2)})\cdots\varphi(x^{(n)})\rangle_{\mathrm{con}}=\frac{\delta^nW}{\delta\varphi(x^{(1)})\ldots\varphi(x^{(n)})}|_{j=0}\,.$$

- $\int D\varphi\ldots$ only formal, '$\mathbb{R}^4$-fold' integral, divergences
- LQFT: define as limit of a sequence of rigorously defined finite dimensional integrals
- numerical (=NP) evaluation possible (Monte Carlo)

Standard QFT (via Feynman rules) goes like this:

- define $\int D\varphi\ldots$ for the Gaussian case $g_0=0$ [Wiener integral]
- for $g_0>0$ Taylor expand $\mathrm{e}^{-\frac{g_0}{4!}\int d^4x\varphi^4}$in $g_0$
- $\Rightarrow$only moments of Gauss integrals required $\left[\int dx\,\mathrm{e}^{-\frac{1}{2}x^2}x^n\right]$

  $\rightarrow$Feynman rules + renormalization algorithm

critique:

- definition and approximation mixed up
- there is no well-defined object, to which we *then* apply an approximation

## 1.2 A little bit of renormalization

Fourier: $\tilde{\varphi}(p) = \int d^4x \, \mathrm{e}^{-ipx} \varphi(x)$
naive:

$$\langle \tilde{\varphi}(p)\tilde{\varphi}(q)\rangle = (2\pi)^4 \delta^4(p+q) \, \frac{1}{p^2+m_0^2} \, \{1 + g_0 \times \infty\}$$

$\Longrightarrow$ regularize, for *example*

$$\frac{1}{p^2+m_0^2} \rightarrow \frac{\mathrm{e}^{-p^2/\Lambda^2}}{p^2+m_0^2} \quad \Longleftrightarrow$$

$$S_\Lambda[\varphi] = \int d^4x \left\{ \frac{1}{2} \, \varphi(-\partial^2 + m_0^2) \mathrm{e}^{\partial^2/\Lambda^2} \varphi + \frac{g_0}{4!} \, \varphi^4 \right\}$$

$\Longrightarrow$ now correlations are finite
match asymptotically for $p^2 \rightarrow 0$

$$\langle \tilde{\varphi}(p)\tilde{\varphi}(q)\rangle \simeq (2\pi)^4 \delta(p+q) \, \frac{Z}{p^2+m_R^2} \simeq (2\pi)^4 \delta(p+q) \frac{Z}{m_R^2}\left[1 - \frac{p^2}{m_R^2} + \ldots\right]$$

to fix $Z, m_R$ by the 'fit'

- read off $Z(g_0, m_0/\Lambda)$ and $m_R/\Lambda = f(g_0, m_0/\Lambda)$ [series in $g_0$ !]

similarly (sketchy!)

$$\langle \tilde{\varphi}\tilde{\varphi}\tilde{\varphi}\tilde{\varphi} \rangle_{\mathrm{con}} = Z^2 (\mathrm{factors}) \, g_R$$

- $g_R = g_0 h(g_0, m_0/\Lambda)$ measure of some scattering strength
- $\propto g_0 \Leftrightarrow$ vanishes in the Gaussian case

Renormalizability:

- all correlations of $Z^{-1/2}\tilde{\varphi}$ expressed as functions of $m_R, g_R$ exist for $\Lambda/m_R \rightarrow \infty$
- more precisely: only the expansion coefficients in $g_R$ exist!
- independent of regularization details $\leftrightarrow$ universality

We usually assume (and check numerically) that such structural properties proven to all orders (!!) also hold NP [$\leftrightarrow$ lattice, see below].

## 1.3 Why PT is insufficient

The above approach cannot work if

- the spectrum of the free theory=zeroth order is *qualitatively* wrong example: confined quarks
- often one shows *within PT*: bound state masses $\propto e^{-c/g_R}$
- $\Rightarrow$ invisible in any order PT, $c$ is *uncomputable* in PT

Even if this is not so: convergence of the series?

We discuss a toy example

$$Z(\lambda) = \int_{-\infty}^{\infty} d\phi e^{-\phi^2 - \frac{\lambda}{4!}\phi^4}$$

Exact answer, $K_{1/4}$ is a Bessel function (`besselk(1/4,.)` in `Matlab`):

$$Z(\lambda) = \sqrt{6/\lambda}\, e^{3/\lambda} K_{1/4}(3/\lambda)$$

All order PT:

$$c_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} Z(\lambda)|_{\lambda=0} = \frac{1}{n!} \int_{-\infty}^{\infty} d\phi e^{-\phi^2} \phi^{4n} = \frac{\Gamma(2n+1/2)}{\Gamma(n+1)}$$

Like all PT: asymptotic expansion:

- radius of convergence zero
- Dyson argument: must be so, since $Z(\lambda), \lambda < 0$ non existing
- but: asymptotic series

$$\left| Z(\lambda) - \sum_{n=0}^{N} c_n \lambda^n \right| \propto \lambda^{N+1} \quad \text{as} \quad \lambda \searrow 0$$

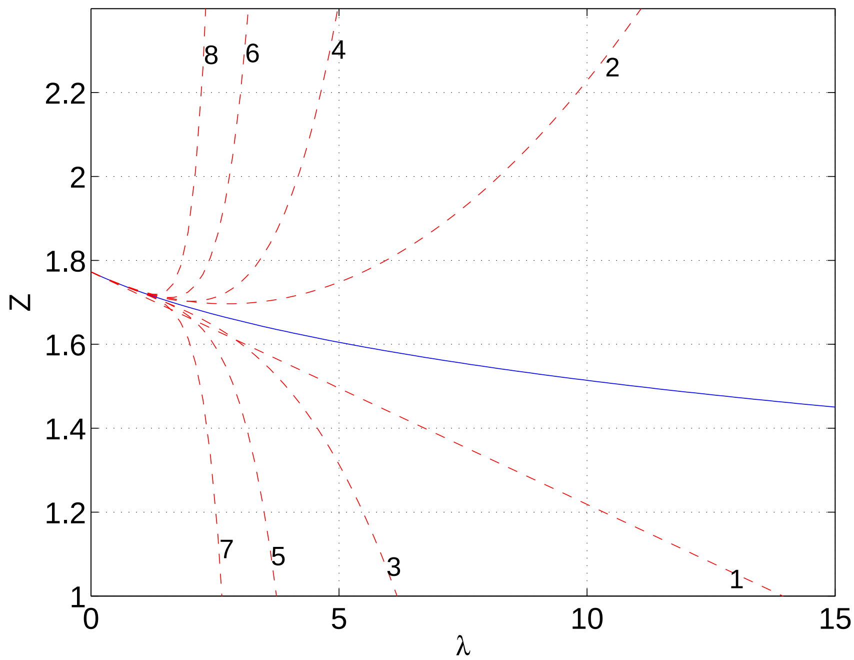


**Figure 1.** Perturbative series at various truncation orders.

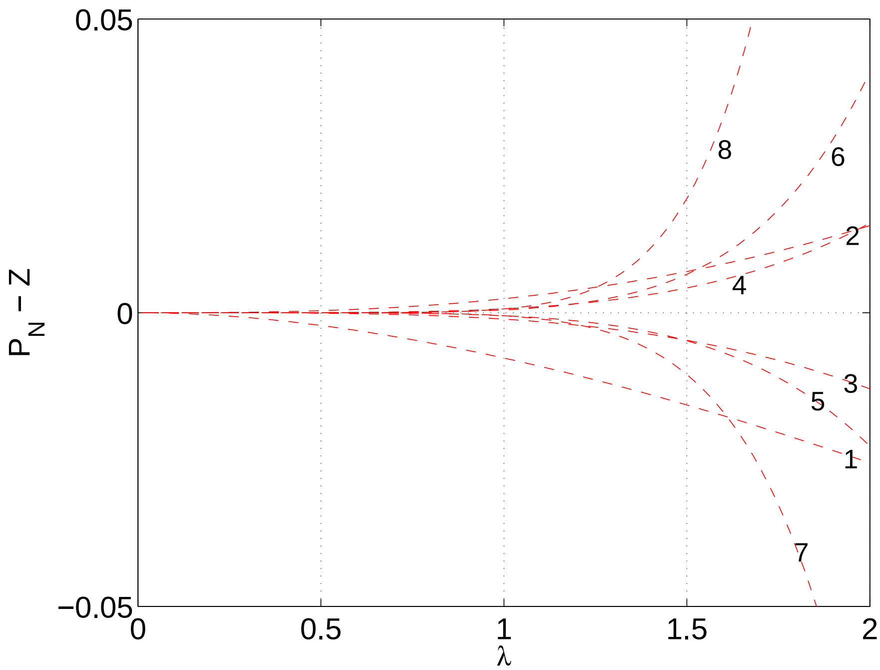


**Figure 2.** Difference between PT and exact.

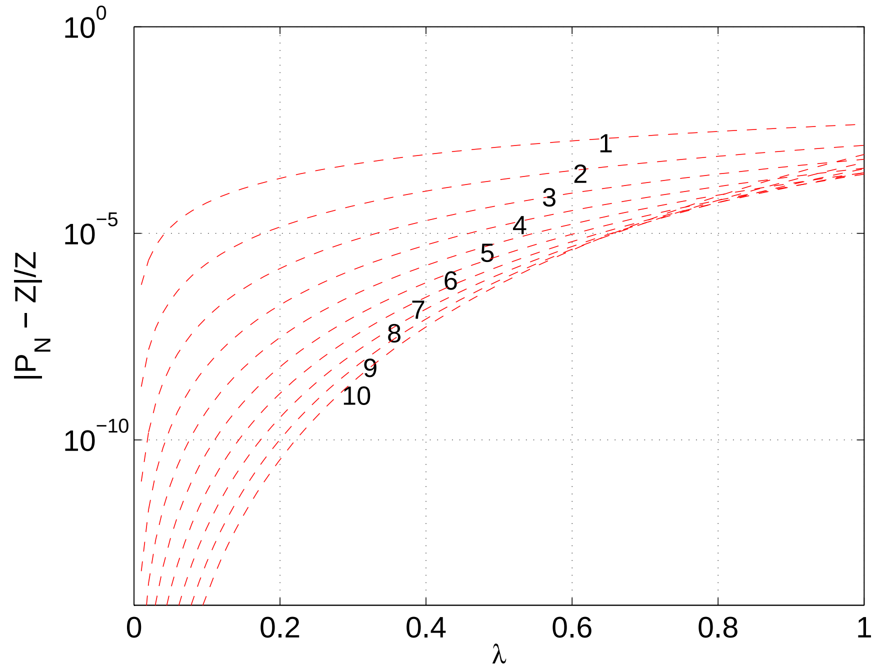


**Figure 3.** Relative approximation error at small $\lambda$: loops make precision!

- at $\lambda < 0.5$: PT precise, high orders good (QED like)
- at $\lambda \approx 2$: 3-4 'loop' optimal (1%), higher worse (QCD, high energy)
- at $\lambda \approx 5$: 1 'loop' optimal (10%), higher orders useless (QCD, tau mass?)
- at $\lambda > 12\,[\lambda/4! > 1/2]$: PT hopeless though $Z(\lambda)$ smooth and boring..

# 2 Lecture: Elementary lattice technology

We are motivated to try to formulate NP QFT....

## 2.1 Discretized space time

- grid already for PDEs
- reasonable: never $\infty$ resolution

$\Longrightarrow \varphi(x)$ not for $x \in \mathbb{R}^4$ but only $x \in (a\mathbb{Z})^4$, hypercubic lattice with spacing $a$:

$$x_\mu = a n_\mu, \quad n_\mu \,\text{integer}, \quad [a] = \text{length}$$

obvious idea:

$$\int D\varphi .... \longrightarrow \prod_x \int_{-\infty}^{\infty} d\varphi(x) ....,$$

- better, but still infinite-fold
- still not obviously rigorous and still not for Computers

## 2.2 Classical continuum limit

- what becomes of $S[\varphi] = \int d^4x ....?$
- first: discretized classical field theory
- imagine: $\varphi$ still exists on $\mathbb{R}^4$, approximate by only using values on $(a\mathbb{Z})^4$

$$\partial_\mu \varphi(x) \rightarrow \partial_\mu^{\text{lat}} \varphi(x) = \frac{1}{a}\left[\varphi(x + a\hat{\mu}) - \varphi(x)\right]$$

or

$$\partial_\mu^{*\,\mathrm{lat}}\varphi(x) = \frac{1}{a}\,[\varphi(x) - \varphi(x - a\hat{\mu})],$$

$$\hat{0} = (1,0,0,0), \quad \hat{1} = (0,1,0,0), \quad \ldots.$$

$\Longrightarrow$

$$\partial_\mu^{\mathrm{lat}}\varphi(x) = \partial_\mu\varphi(x) + \mathrm{O}(a), \quad \partial_\mu^{\mathrm{lat}}\varphi(x) = \partial_\mu\varphi(x + a\hat{\mu}/2) + \mathrm{O}(a^2)$$

$$S^{\mathrm{lat}} = a^4 \sum_x \left\{ \frac{1}{2}\,(\partial_\mu^{\mathrm{lat}}\varphi)^2 + \frac{m_0^2}{2}\,\varphi^2 + \frac{g_0}{4!}\,\varphi^4 \right\} \approx S$$

partial ‘integration’ ($\varphi, \chi$ must decay at $\infty$):

$$a^4 \sum_x \varphi \partial_\mu^{\mathrm{lat}} \chi = -\, a^4 \sum_x (\partial_\mu^{*\,\mathrm{lat}}\varphi)\chi$$

We now drop ‘lat’ [clear from context]
symmetrized standard Laplacian:

$$\partial_\mu \partial_\mu^* \varphi(x) = \partial_\mu^* \partial_\mu \varphi(x) = \sum_\mu [\varphi(x + a\hat{\mu}) + \varphi(x - a\hat{\mu}) - 2\varphi(x)]$$

Discretized action $\rightarrow$ classical field eqn. $(-\,\partial_\mu^*\partial_\mu + m_0^2)\varphi + (g_0/6)\varphi^3 = 0$

This may be generalized to Maxwell theory for example to compute cavities...

## 2.3 Finite volume

Periodic boundary conditions (and $a > 0$), demand

$$\varphi(x \pm L\hat{\mu}) = \varphi(x), \quad L/a \,\text{integer}$$

- only $(L/a)^4$ independent points, $\{\varphi(x)\} \leftrightarrow \mathbb{R}^{(L/a)^4}$
- now $a^4 \sum_x \ldots.$ finite sum (in $S$)

$$\int D\varphi\ldots. \longrightarrow \prod_x \int_{-\infty}^{\infty} d\varphi(x)\ldots.$$

rigorously defined $(L/a)^4$-fold ordinary integral.

We may label

$$x_\mu = 0, a, 2a, \ldots, L - a$$

or *equivalently* (for even $L/a$)

$$x_\mu = -\frac{L}{2}, -\frac{L}{2}+a, \dots, 0, \dots, \frac{L}{2}-a$$

The same points appear in a different order! Asymm. only apparent: $\frac{L}{2} \triangleq -\frac{L}{2}$

Symmetries:

- discrete translations
- O(4,$\mathbb{Z}$) hypercubic subgroup of O(4,$\mathbb{R}$)
  - rotations by $\pi/2$ through any plane
  - reflections
  - $2 \times 192$ elements
  - restoration of O(4,$\mathbb{R}$) in continuum limit (later)

## 2.4 Lattice Fourier

One direction and $L = \infty$ at first:

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} \mathrm{e}^{ip(x-y)} = \delta(x-y) \longrightarrow \int_{-\pi/a}^{\pi/a} \frac{dp}{2\pi} \mathrm{e}^{ip(x-y)} = \frac{1}{a}\delta_{x,y}$$

- Brillouin zone $[-\pi/a, \pi/a]$, $2\pi/a$ periodic
- $a\sum_x \frac{1}{a}\delta_{x,y} = 1 = \int dx\, \delta(x-y)$
- smallest $\Delta x = a \leftrightarrow |p| \leqslant \pi/a$ cutoff

Now finite $L$:

$$\frac{1}{L}\sum_p \mathrm{e}^{ip(x-y)} = \frac{1}{a}\delta_{x,y}^{\mathrm{per}} \quad \text{with} \quad p \in \frac{2\pi}{L}\mathbb{Z}$$

for example

$$p = 0, \frac{2\pi}{L}, 2\frac{2\pi}{L}, \dots, (L/a-1)\frac{2\pi}{L} = \frac{2\pi}{a} - \frac{2\pi}{L}$$

- $\delta_{x,y}^{\mathrm{per}} = \delta_{x,y\pm L}^{\mathrm{per}}$ $\rightarrow$ 'per' dropped again
- $D = 4$ each component $p_\mu$ as above
- $\delta_{x,y} = 1$ if $x_\mu = y_\mu \bmod L$ for all $\mu$

$$\tilde{f}(p) = a^4 \sum_x \mathrm{e}^{-ip\cdot x} f(x), \qquad f(x) = \frac{1}{L^4}\sum_p \mathrm{e}^{ip\cdot x}\tilde{f}(p)$$

summary:

- UV cutoff: $x_\mu$ discrete (steps $a>0$)
- IR cutoff: $p_\mu$ discrete $(2\pi/L>0)$
- UV+IR: everything discrete and finite, limits can then be investigated

## 2.5 The lattice path integral

Obvious proposal now:

$$\mathrm{e}^{W[j]} = \frac{1}{Z}\left[\prod_x \int d\varphi(x)\right] \mathrm{e}^{-S[\varphi]+a^4\sum_x j(x)\varphi(x)}$$

correlations: $\frac{\delta}{\delta j(x)} \to \frac{1}{a^4}\frac{\partial}{\partial j(x)}$

Dimensions, dimensionless: $a\varphi, am_0, L/a, g_0 \;\Rightarrow$

$$a^n \langle \varphi(0)\varphi(x^{(2)})\cdots\varphi(x^{(n)})\rangle = f(x^{(i)}/a, am_0, g_0, L/a)$$

- these are well defined bare correlations
- may or may not be expanded in $g_0$
- usually $L/a \to \infty$ exists at fixed $x^{(i)}/a, am_0, g_0$
- Computer: at large $L/a$ insensitive to value (see below)
- continuum limit ($\triangleq$ renormalization) more complicated:

*match*:

$$L^{-4}\langle \tilde{\varphi}(p)\tilde{\varphi}(-p)\rangle = \frac{Z}{\hat{p}^2 + m_R^2}$$

at $p=0$ and $p=p_* = (2\pi/L, 0, 0, 0)$ $[\hat{p}_*^2 \approx p_*^2]$ to obtain

$$Z = Z(am_0, g_0, L/a), \quad am_R = f(am_0, g_0, L/a)$$

One may show now (Fourier)

$$\ln\left[a^2\langle \varphi(0)\varphi(x)\rangle\right] \simeq -m_R\,|x| \quad \text{at large} \quad |x|$$

- $\xi = m_R^{-1}$ is a correlation length exposed in physical correlations

- scaling region (continuum, UV cutoff limit) is reached if we tune $am_0$ such that $a/\xi = am_R \ll 1$ holds (for some $g_0$)
- thermodynamic region ( $\sim$ infinite volume, IR cutoff limit): $L/\xi = Lm_R \gg 1$

Renormalized coupling (sketchy):

$$a^{-12}\langle\tilde{\varphi}\tilde{\varphi}\tilde{\varphi}\tilde{\varphi}\rangle_{\mathrm{con}} = Z^2(\mathrm{factors})\, g_R$$

Now all correlations are conjectured to have a NP UV+IR limit

$$Z^{-n/2}\langle\varphi(x^{(1)})\varphi(x^{(2)})\cdots\varphi(x^{(n)})\rangle = m_R^n f(m_R x^{(i)}; g_R) \times [1 + \underbrace{\mathrm{O}(a^2 m_R^2)}_{\mathrm{Symanzik}} + \underbrace{\mathrm{O}(\mathrm{e}^{-cm_R L})}_{L\ddot{u}\mathrm{scher}}]$$

- we may or may not expand in $g_R$
- if we do we may or may not approximate the true answer (c.f. sect. 1.3)
- really: dimensionless numbers only, lattice $a$ for 'book keeping' only
- really: the same for $Z$: universal $\leftrightarrow$ physics $\leftrightarrow$ ratios where $Z$ drops out
- if we know $am_R = \#$ and $m_R$ is identified with a mass in nature $\Rightarrow a$ may be quoted in $\mathrm{GeV}^{-1}$ for a given set of lattice parameters

# 3 Lecture: QCD discretized on a lattice

Non Abelian gauge theory was formulated on discretized space time by Wilson.

- geometric structure needs to be taken into account to have
- exact gauge invariance *in the regularized theory*
- considered crucial for renormalizability and universality
- unlike continuum we will need no gauge fixing
- which is very problematic beyond PT...

## 3.1 Gauge invariance

Consider a complex matter field $\psi(x)$ with $N$ components that transforms under a *local* $\mathrm{SU}(N)$:

$$\psi(x) \rightarrow h(x)\psi(x), \quad h(x) \in \mathrm{SU}(N)$$

In the continuum, a gauge field is a 'device' to construct a covariant derivative;

$$D_\mu\psi(x) \equiv (\partial_\mu + A_\mu(x))\psi(x) \rightarrow (\partial_\mu + A'_\mu(x))h(x)\psi(x) = (!)h(x)D_\mu\psi(x)$$

- including $A \rightarrow A'$, $D_\mu\psi$ transforms like $\psi$
- requires $A'_\mu = hA_\mu h^{-1} - (\partial_\mu h)h^{-1} = hA_\mu h^{-1} + h\partial_\mu h^{-1}$

equivalent (1st order in $\varepsilon$):

$$\varepsilon_\mu D_\mu\psi(x) \simeq \exp(\varepsilon_\mu A_\mu) \times \psi(x+\varepsilon) - \psi(x)$$

- parallel transport $x \leftarrow x+\varepsilon$ with $1 \approx \exp(\varepsilon_\mu A_\mu) \in \mathrm{SU}(N)$ before comparing
- infinitesimal transformation $\Rightarrow A_\mu \in$ Lie algebra
- $\exp(\varepsilon_\mu A'_\mu) \approx h(x)\exp(\varepsilon_\mu A_\mu)h^{-1}(x+\varepsilon)$: eats $h(x+\varepsilon)$, spits out $h(x)$

This is the clue for derivative $\longrightarrow$ difference, lattice covariant derivative:

$$D_\mu\psi(x) = U(x,\mu)\psi(x+a\hat{\mu}) - \psi(x)$$

$$D^*_\mu\psi(x) = \psi(x) - U(x-a\hat{\mu},\mu)^{-1}\,\psi(x-a\hat{\mu})$$

- $U(x,\mu) \rightarrow h(x)U(x,\mu)h^{-1}(x+a\hat{\mu})$
- *finite* transport, $U(x,\mu) \in$ Lie *group* $\mathrm{SU}(N)$ $[U^{-1} = U^\dagger, \det(U) = 1]$
- $U(x,\mu)$ lives on the *links* of the lattice
- configuration $\equiv$ set of $4 \times (L/a)^4$ $\mathrm{SU}(N)$ matrices

## 3.2 Wilson Yang-Mills action

Matter part of the action gauge invariant by using *co*variant differences, but $\rightarrow$ need invariant action for $U(x,\mu)$ itself

- $S[\varphi]$ suppresses variation via $(\partial_\mu\varphi)^2$
- gauge actions must suppress curvature

continuum: $[D_\mu, D_\nu]\psi = F_{\mu\nu}\psi \rightarrow -\operatorname{tr}(F_{\mu\nu})^2$, $\quad F_{\mu\nu}(x) \rightarrow h(x)F_{\mu\nu}(x)h^{-1}(x)$

2 paths: $x \leftarrow x+a\hat{\mu} \leftarrow x+a\hat{\mu}+a\hat{\nu}$ and $x \leftarrow x+a\hat{\nu} \leftarrow x+a\hat{\mu}+a\hat{\nu}$ (picture!)

transporters: $U(x,\mu)U(x+a\hat{\mu},\nu)-U(x,\nu)U(x+a\hat{\nu},\mu)=M\neq 0 \leftrightarrow$ curvature around plaquette $(x,\mu,\nu)$.

$$MM^\dagger = 2 - U_\Box(x,\mu,\nu) - U_\Box^\dagger(x,\mu,\nu) \quad 0\leqslant \mathrm{tr}(MM^\dagger)=0 \Leftrightarrow M=0$$

$$U_{\mathrm{pl}}(x,\mu,\nu)=U(x,\mu)U(x+a\hat{\mu},\nu)U^\dagger(x+a\hat{\nu},\mu)U^\dagger(x,\nu)$$

gauge behavior:

$$U_\Box(x,\mu,\nu)\to h(x)U_\Box(x,\mu,\nu)h^{-1}(x)$$

Wilson action:

$$S[U]=\frac{\beta}{2N}\sum_{x,\mu\neq\nu}\mathrm{tr}[1-U_\Box(x,\mu,\nu)]\geqslant 0$$

equivalent:

$$S[U]=\beta\sum_{x,\mu<\nu}\left[1-\frac{1}{N}\mathrm{Re}\,\mathrm{tr}\,U_\Box\right]$$

One can demonstrate the classical continuum limit:

- $A_\mu(x)$ given in $\mathbb{R}^4$, $U(x,\mu)=\exp[a\,A_\mu(x)]$ on $x\in(a\mathbb{Z})^4$:

$$S[U]\approx -\frac{\beta}{2N}a^4\sum_{x,\mu,\nu}\frac{1}{2}\mathrm{tr}\,(F_{\mu\nu})^2\approx -\frac{1}{2g^2}\int d^4x\,\mathrm{tr}\,(F_{\mu\nu})^2 \quad \text{with} \quad \beta=\frac{2N}{g^2}$$

## 3.3 Dirac Wilson fermions

Euclidean fermions in the continuum:

$$S=\int d^4x\overline{\psi}\,(\gamma_\mu D_\mu+m_0)\psi$$

$$\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}, \quad \gamma_\mu^\dagger=\gamma_\mu.$$

Obvious discretization, 'naive' lattice fermions

$$S_{\mathrm{naive}}=a^4\sum_x\overline{\psi}\,(\gamma_\mu\tilde{D}_\mu+m_0)\psi$$

with symmetrized (antihermitean) derivative

$$\tilde{D}_\mu\psi(x)=\frac{1}{2}(D_\mu+D_\mu^*)\psi(x)=\frac{1}{2a}\big[U(x,\mu)\psi(x+a\hat{\mu})-U^{-1}(x-a\hat{\mu},\mu)\psi(x-a\hat{\mu})\big]$$

Problem: species doubling. Free case, Fourier expansion

$$[\gamma_\mu \tilde{\partial}_\mu + m_0]\, u e^{ip\cdot x} = [i\gamma_\mu \tilde{p}_\mu + m_0]\, u e^{ip\cdot x}, \quad \tilde{p}_\mu = \frac{1}{a}\sin(a p_\mu)$$

- when all components have $a p_\mu \ll 1 \;\Rightarrow \tilde{p}_\mu \approx p_\mu \;\leftrightarrow$ classical continuum limit
- but also if $a p_\mu = \pi - a q_\mu$ with $a q_\mu \ll 1$ $\tilde{p}_\mu \approx q_\mu$
- not one area in Brillouin zone, but $2^4 = 16$ $\Rightarrow$ the continuum limit has 16 degenerate 'flavors'

The Wilson term cures this:

$$S_W = a^4 \sum_x \overline{\psi}\,(\gamma_\mu \tilde{D}_\mu + m_0 - \frac{a\,r}{2} D_\mu D_\mu^*)\psi$$

- extra term $=\mathrm{O}(a)$ distortion if all $a p_\mu \ll 1$, but suppresses 'doubler' modes
- clean spectrum
- but no more chiral symmetry (distinguishing zero mass):

$$\left\{\gamma_5, \gamma_\mu \tilde{D}_\mu + m_0 - \frac{a\,r}{2} D_\mu D_\mu^*\right\} = 0 \Leftrightarrow m_0 = 0 \;\text{true only for}\; r = 0$$

- the reason is 'deep' (Nielsen Ninomiya no go theorem)
- alternative discretizations: staggered, twisted mass and others
- Ginsparg Wilson: Modified chiral symmetry on the lattice ($\mathrm{O}(a)$ extra terms in transformations)
- No NP formulation exists for chiral gauge theories (and thus the complete Standard Model) so far! [*chiral* gauge invariance crucial for renormalizability]

## 3.4 Fermion simulation

- $\psi(x), \overline{\psi}(x)$ are Grassmann valued
- they must be integrated exactly using:

$$\int \prod_{x,\alpha} d\psi_\alpha d\overline{\psi}_\alpha\, e^{-a^8 \sum_{x,y} \overline{\psi}(x) A(x,y)\psi(y) + a^4 \sum_x (\bar{j}\psi - \overline{\psi} j)} = \det(A)\, e^{-a^8 \sum_{x,y} \bar{j}(x) A^{-1}(x,y) j(y)}$$

For currents $J_\Gamma = \overline{\psi}\,\Gamma\psi$ for example

$$\langle J_\Gamma(x) J_\Gamma(y)\rangle = \frac{1}{Z}\int DU\,\det(A[U])\mathrm{e}^{-S[U]}\big[-\mathrm{tr}(\Gamma A^{-1}(x,y)\Gamma A^{-1}(y,x)\Gamma + \text{discon.}\big]$$

- Monte Carlo with effective Boltzmann $\det(A[U])\mathrm{e}^{-S[U]}$ in $U$ only
- nonlocal funnction('al') of $U(x,\mu)$
- well developed (Pseudofermions, Hybrid Monte Carlo), but costly!

# 4 A few problems

## 4.1 Two point function and spectrum

Some background first. You should know (or believe for the moment) the Feynman-Kac formula in *quantum mechanics*. A Hamiltonian

$$\hat{H} = \frac{1}{2}\hat{p}^2 + V(\hat{x})$$

is related to the Euclidean path integral over periodic orbits by

$$Z = \mathrm{tr}\,\mathrm{e}^{-T\hat{H}} = \int_{x(0)=x(T)} Dx(t)\,\mathrm{e}^{-S[x(t)]} = \sum_n \mathrm{e}^{-TE_n}$$

with

$$S = \int_0^T dt\left\{\frac{1}{2}\dot{x}^2 + V(x)\right\}$$

and

$$\langle x(0)x(t)\rangle = \frac{1}{Z}\int Dx\mathrm{e}^{-S}x(0)x(t) = \frac{1}{Z}\sum_{m,n}|\langle m|\hat{x}\,|n\rangle|^2\,\mathrm{e}^{-(T-t)E_m - tE_n}$$

for the two point correlation.

If the Euclidean time extent is much larger than the inverse gap $T(E_1 - E_0) \gg 1$ and $t \ll T$, then we isolate the ground state and the two point function knows the excitation spectrum

$$\langle x(0)x(t)\rangle \simeq \sum_n |\langle 0|\hat{x}\,|n\rangle|^2\mathrm{e}^{-t(E_n - E_0)}.$$

In LQFT these things are very similar in the time-momentum form

$$a^3 \sum_{\vec{x}} \mathrm{e}^{-i\vec{p}\cdot\vec{x}} \langle \varphi(0)\varphi(x) \rangle = \Delta(x_0, \vec{p}\,) \propto \mathrm{e}^{-tE(\vec{p}\,)}$$

where $E(\vec{p}\,)$ is the gap in the channel of momentum $\vec{p}$, the energy of the lowest state *relative to the vacuum* as required in QFT. In particular at $\vec{p} = 0$, $E(\vec{0}) = m_R$ is the energy of the lightest particle at rest, that can be excited from the vacuum by the field *operator* $\hat{\varphi}$ analogous to $\hat{x}$ in QM.

- derive the propagator in momentum space for a free scalar particle

$$a^4 \sum_{\vec{x}} \mathrm{e}^{-ip\cdot x} \langle \varphi(0)\varphi(x) \rangle = \frac{1}{\hat{p}^2 + m_0^2}$$

- Fourier transform to $\Delta(x_0, \vec{p}\,)$ for an infinite lattice and compute $E(\vec{p}\,)$. Hint: Consider the $p_0$ integration as a line integral in $\mathbb{C}$ and close the contour.
- Do you find $m_R^2 = m_0^2$ before and/or after taking the continuum limit?
- What about the dispersion relation $\sqrt{\vec{p}^{\,2} + m^2}$?
- What is the momentum space Dirac Wilson propagator? Consider again computing $m_R$ for the usual choice $r = 1$.

## 4.2 Invariant group measure, Monte Carlo

The gauge links are integrated over SU($N$). To have a gauge invariant theory the measure must be invariant under group multiplications by $g$, $h \in$ SU($N$) from either side,

$$I[f] = \int_{\mathrm{SU}(N)} dU f(U) = \int_{\mathrm{SU}(N)} dU f(gUh)$$

has to hold for any function $f$ on SU($N$). We demand it to be normalized, $I = 1$ if $f \equiv 1$.

- To get the idea, discuss the analogous (but trivial) case of invariant summation over an arbitrary *finite* group.

We now specialize to $N = 2$. There is the quaternionic parameterization

$$U(u) = u_0 + i\,\vec{u} \cdot \vec{\tau}\,, \quad u_0^2 + \vec{u}^{\,2} = 1 = |u|^2$$

which identifies SU(2) with the sphere $S_3$ in Euclidean four space.

- Argue that

$$\frac{1}{2\pi^2}\int d^4u\,\delta(|u|-1)f(U(u))$$

  is a correct way to write the invariant integration.
  Hint: $U\to gUh$ corresponds to an SO(4) rotation of $(u_0,\vec{u})$.

- In the end, we need to integrate by Monte Carlo. For a Metropolis update, we want to *propose*, at a given 'old' $U$, a move $U\to UV$ with a random $V\in \mathrm{SU}(2)$. This has to be generated such that
  - $V$ and $V^{-1}$ are proposed with the same probability
  - there is a control parameter that limits the distance of $V$ from the unit element (to have a decent acceptance rate).

- Given a flat perfect random number generator, design an algorithm for $V$

- Discuss how, if $U\in \mathrm{SU}(3)$, your proposal can be embedded in SU(3) via two or more SU(2) subgroups, such that *all* SU(3) elements can be reached under *multiple* moves. This is how many updates in quenched QCD work in principle.

## 4.3 Confinement at strong coupling

Wilson loop: A closed loop on the lattice is given by a cyclic sequence of points $\mathcal{C}=\{x_1,\,x_2,\,\ldots,\,x_n\}$ such that $(x_i,\,x_{i+1})$ are nearest neigbors (cyclic: $x_{n+1}=x_1$). Setting $x_{i+1}=x_i+a\hat{\mu}_i$ the Wilson loop observable is

$$W(\mathcal{C})=\mathrm{tr}[U(x_1,\mu_1)U(x_2,\mu_2)\cdots U(x_n,\mu_n)]$$

- Convince yourself that $W$ is gauge invariant

For a rectangular loop $\mathcal{C}_{t,l}$ of size $t\times l$ in a plane one can show that

$$\langle W(\mathcal{C}_{t,l})\rangle\propto \mathrm{e}^{-tV(l)}\quad (t\to\infty)$$

holds where $V(l)$ is a definition of the potential between heavy quarks due to their interaction with the gluons. One confinement criterion is the linear rise

$$\text{confinement}\leftrightarrow V(l)\propto\sigma\times l\quad (l\to\infty)$$

at large distance, in a way $\sigma$ is the gauge-analog of a mass gap.

- Compute $Z = \int \prod_{x,\mu} dU(x,\mu) \mathrm{e}^{-S[U]}$ as an expansion in $\beta$ (strong coupling expansion) to leading nontrivial order
- to do this prove first (for SU($N$), $N \geqslant 3$)

$$I_{\alpha\beta\gamma\delta} = \int dU U_{\alpha\beta} U^*_{\gamma\delta} = \frac{1}{N} \delta_{\alpha\delta} \delta_{\beta\gamma}, \qquad \int dU U_{\alpha\beta} U_{\gamma\delta} = 0 = \int dU U_{\alpha\beta}$$

  hints:
  - use the left and right invariance of the measure
  - $\delta_{\alpha\beta}$ is the only SU($N$) invariant tensor to build $I_{\alpha\beta\gamma\delta}$
- derive the average plaquette $\frac{1}{N}\langle \mathrm{Re}\, \mathrm{tr}\, U_\Box \rangle$ in this approximation
- now compute $\langle W(\mathcal{C}_{t,l}) \rangle$ to leading order and prove confinement at small $\beta$
- what is $a^2\sigma$ in this approximation?

Note that: a) small $\beta$ is far from the continuum limit, hence the above is qualitative at best b) confinement is a very natural situation in Lattice Yang Mills

- prove rigorously that confinement persists in the continuum limit $\beta \to \infty$ [this will win you a prestigeous prize!]

# 5 Further study

The lattice formulation of QFT has been intensely investigated for several decades now. This has led to a number of textbooks on the subject. The ones known to the author are:

- Ref. [1] by one of the founding fathers of the field who performed first simulations is rather old by now. Nevertheless it can still serve as an easily readable first introduction.
- Real text books are [2],[3],[4],[5],[6] and one should have a look, which style and content suits one.
- [7] is a QFT book with the connection to critical phenomena in view. Its volume is impressive.
- [8] and [9] are lecture notes of courses previously taught at HU

In particular in the books one may find a lot more references.

# Bibliography


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