Aiming for a higher precision in the QCD Λ-parameter using running couplings in the three-flavour theory

Patrick Fritzsch

Humboldt-Universität zu Berlin

Westfälische-Wilhelms Universität, Münster

May 26$^{\mathrm{th}}$, 2014

MOTIVATION

The strength of fundamental interactions

Knowing α allows to test the consistency of the underlying theory und thus our understanding of the SM.

$$\alpha(\mu) = \bar{g}^2(\mu)/4\pi$$

How to estimate α?

- choose a suitable *experimental* observable $O(\mu)$
- having a decent perturbative behaviour $\quad O(\mu) = c_1\alpha(\mu) + c_2\alpha^2(\mu) + \ldots$
- determine $\alpha(\mu)$ by *matching* theory and experiment

The strength of fundamental interactions

Knowing α allows to test the consistency of the underlying theory und thus our understanding of the SM.

$$\alpha(\mu) = \bar{g}^2(\mu)/4\pi$$

How to estimate α?

- choose a suitable *experimental* observable $O(\mu)$
- having a decent perturbative behaviour $O(\mu) = c_1\alpha(\mu) + c_2\alpha^2(\mu) + \ldots$
- determine $\alpha(\mu)$ by *matching* theory and experiment

QED

$$(g_e - 2)_{\rm exp} \; : \quad \alpha_{\rm em} = 7.2973525698(24) \times 10^{-3}$$

$$\text{recoil-velocity meas.} \; : \quad \alpha_{\rm em} = 7.297352585(48) \times 10^{-3}$$

Many factors limit its precision

- Nature (QED$\subset$SM): additional contributions from muon, tau, the weak interactions and hadrons
- uncertainties and systematic errors from experiment & theory

However, very good relative agreement between experiment and theory: $\mathrm{O}(10^{-10})$

$\Rightarrow$ strong limitations to

- a possible structure of the electron, existence of new dark matter particles, ...

The strength of fundamental interactions

Knowing α allows to test the consistency of the underlying theory und thus our understanding of the SM.

$$\alpha(\mu) = \bar{g}^2(\mu)/4\pi$$

How to estimate α?

- choose a suitable *experimental* observable $O(\mu)$
- having a decent perturbative behaviour $\quad O(\mu) = c_1\alpha(\mu) + c_2\alpha^2(\mu) + \dots$
- determine $\alpha(\mu)$ by *matching* theory and experiment

QCD

$$\tau \; : \quad \alpha_s(M_Z) = 0.1198(15)$$

$$e^+e^- \; : \quad \alpha_s(M_Z) = 0.1172(37)$$

Non-Abelian structure leading to

- confinement at low energies (where we live)
- asymptotic freedom at high energies (where PT works)

Problems: higher orders in PT, non-perturbative effects, . . .

The strength of fundamental interactions

Knowing α allows to test the consistency of the underlying theory und thus our understanding of the SM.

$$\alpha(\mu) = \bar{g}^2(\mu)/4\pi$$

How to estimate α?

- choose a suitable *experimental* observable $O(\mu)$
- having a decent perturbative behaviour $\quad O(\mu) = c_1\alpha(\mu) + c_2\alpha^2(\mu) + \dots$
- determine $\alpha(\mu)$ by *matching* theory and experiment

QCD

$$\tau \; : \quad \alpha_s(M_Z) = 0.1198(15)$$
$$e^+e^- \; : \quad \alpha_s(M_Z) = 0.1172(37)$$

Non-Abelian structure leading to

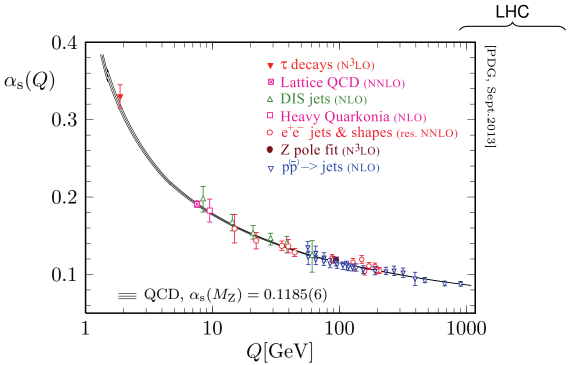
- confinement at low energies (where we live)
- asymptotic freedom at high energies (where PT works)

Problems: higher orders in PT, non-perturbative effects, ...

Non-perturbative definition of running couplings: $\quad \alpha_O(\mu) = O(\mu)/c_1$

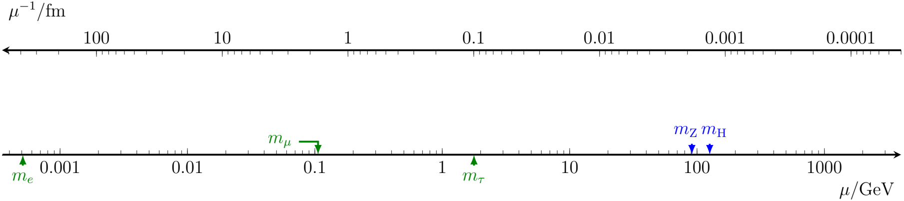
$\alpha_s^{(5)}(M_Z)$ world averages

PARTICLE DATA GROUP	2013
	[Beringer et al.]
w/ lattice:	$0.1185(6)$
w/o lattice:	$0.1183(12)$
FLAVOR LATTICE AVERAGING GROUP	2014
	[1310.8555v2]
lattice only:	$0.1184(12)$



RG runnning of α_s

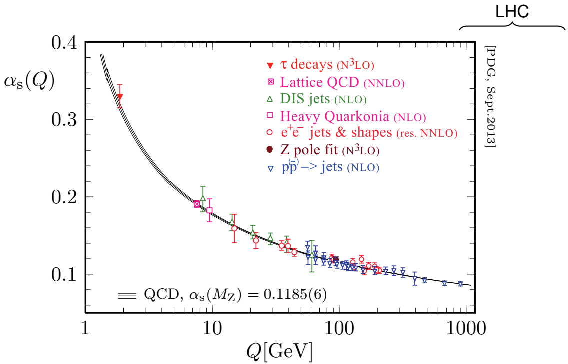
and the landscape of QCD



$\alpha_s^{(5)}(M_Z)$ world averages

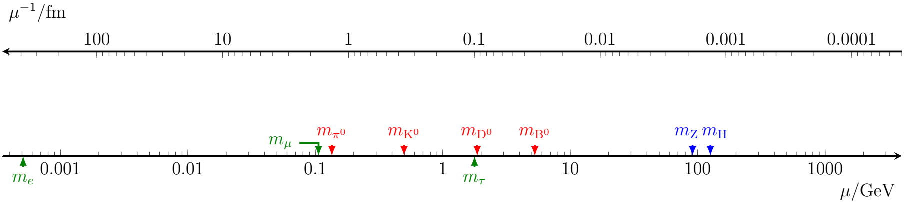
PARTICLE DATA GROUP	2013
	[Beringer et al.]
w/ lattice:	$0.1185(6)$
w/o lattice:	$0.1183(12)$

FLAVOR LATTICE AVERAGING GROUP	2014
	[1310.8555v2]
lattice only:	$0.1184(12)$



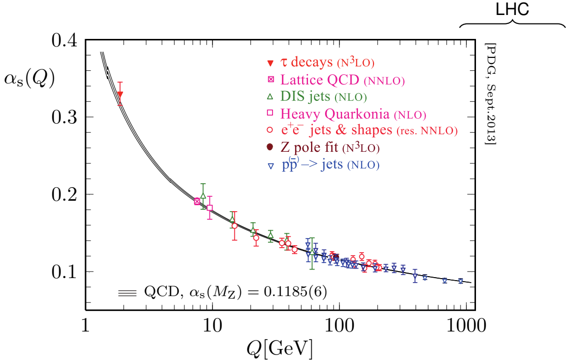
RG runnning of α_s

and the landscape of QCD



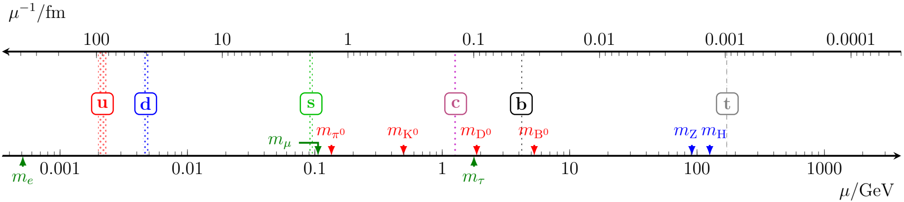
$\alpha_s^{(5)}(M_Z)$ world averages

PARTICLE DATA GROUP	2013
	[Beringer et al.]
w/ lattice:	$0.1185(6)$
w/o lattice:	$0.1183(12)$
FLAVOR LATTICE AVERAGING GROUP	2014
	[1310.8555v2]
lattice only:	$0.1184(12)$



RG runnning of α_s

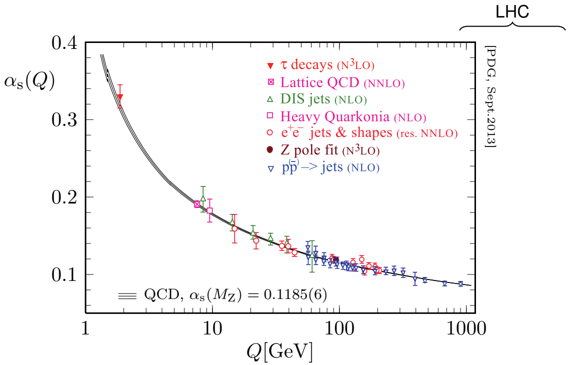
and the landscape of QCD



$\alpha_s^{(5)}(M_Z)$ world averages

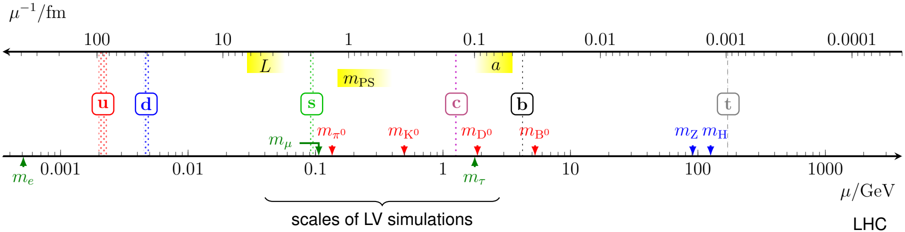
PARTICLE DATA GROUP	2013
	[Beringer et al.]
w/ lattice:	0.1185(6)
w/o lattice:	0.1183(12)

FLAVOR LATTICE AVERAGING GROUP	2014
	[1310.8555v2]
lattice only:	0.1184(12)



RG runnning of α_s

and the landscape of QCD



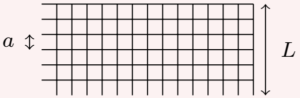
Lattice QCD

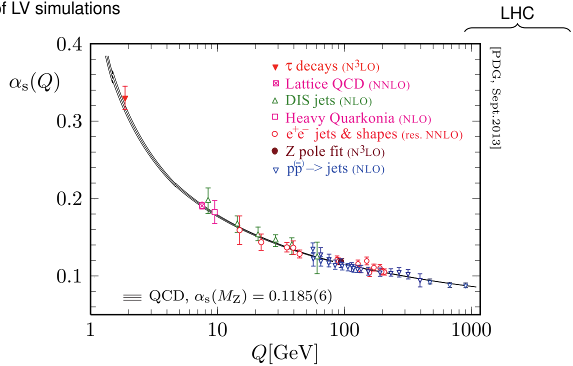
introduces additional scales

UV-cutoff a $\rightsquigarrow$ discretisation effects

IR-cutoff L $\rightsquigarrow$ finite-volume effects

$$L^{-1} \ll \mu_{\min} < Q < \mu_{\max} \ll a^{-1}$$





METHODOLOGY

- RENORMALIZATION GROUP AND Λ-PARAMETER
- FINITE-SIZE SCALING
- LATTICE REGULARISATION
- GRADIENT FLOW
- A NEW RUNNING COUPLING SCHEME

Renormalization Group and Λ–parameter

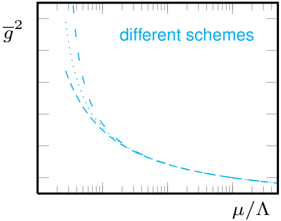
(mass-independent scheme)

RGE:

$$\mu \frac{\partial \bar{g}}{\partial \mu} = \beta(\bar{g}) \quad \overset{\bar{g}\to 0}{\sim} \quad -\bar{g}^3(b_0 + b_1\bar{g}^2 + b_2\bar{g}^4 + \ldots)$$

b_0, b_1 : universal coeff.

$b_{i>1}$: scheme-dependent



Renormalization Group and Λ–parameter

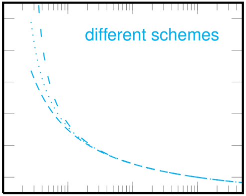
(mass-independent scheme)

RGE:

$$\mu \frac{\partial \bar{g}}{\partial \mu} = \beta(\bar{g}) \quad \overset{\bar{g}\to 0}{\sim} \quad -\bar{g}^3(b_0 + b_1\bar{g}^2 + b_2\bar{g}^4 + \ldots)$$

b_0, b_1 : universal coeff.

$b_{i>1}$: scheme-dependent



Our aim is to compute

$$\Lambda \equiv \mu \left[b_0\bar{g}^2\right]^{-b_1/(2b_0^2)} \mathrm{e}^{-1/(2b_0\bar{g}^2)} \exp\left\{-\int_0^{\bar{g}} \mathrm{d}g \left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\}$$

$\alpha_s(M_Z) = 0.1183(7) \rightsquigarrow \Delta\Lambda/\Lambda \approx 4\%$

- *exact equation* for *mass-independent scheme*
- *trivial scheme dependence:*

$$\Lambda_X/\Lambda_Y = \exp\left(c_{X,Y}/[4\pi b_0]\right), \qquad \alpha_X = \alpha_Y + c_{X,Y}\alpha_Y^2 + \mathcal{O}(\alpha_Y^3)$$

- use a suitable physical coupling (scheme) to compute Λ
  (defined $\forall\mu$, regularisation independent, usual PT behaviour at small α, . . .)

Renormalization Group and Λ–parameter

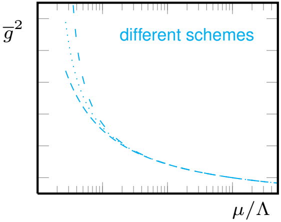
(mass-independent scheme)

RGE:

$$\mu \frac{\partial \bar{g}}{\partial \mu} = \beta(\bar{g}) \quad \overset{\bar{g}\to 0}{\sim} \quad -\bar{g}^3(b_0 + b_1\bar{g}^2 + b_2\bar{g}^4 + \ldots)$$

b_0, b_1 : universal coeff.

$b_{i>1}$: scheme-dependent



Our aim is to compute

$$\Lambda \equiv \mu\left[b_0\bar{g}^2\right]^{-b_1/(2b_0^2)} \mathrm{e}^{-1/(2b_0\bar{g}^2)} \exp\left\{-\int_0^{\bar{g}} \mathrm{d}g \left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\}$$

$\alpha_s(M_Z) = 0.1183(7) \rightsquigarrow \Delta\Lambda/\Lambda \approx 4\%$

- *exact equation* for *mass-independent scheme*
- *trivial scheme dependence:*

$$\Lambda_X/\Lambda_Y = \exp\left(c_{X,Y}/[4\pi b_0]\right), \qquad \alpha_X = \alpha_Y + c_{X,Y}\alpha_Y^2 + \mathcal{O}(\alpha_Y^3)$$

- use a suitable physical coupling (scheme) to compute Λ
  (defined $\forall\mu$, regularisation independent, usual PT behaviour at small α, ...)

which requires non-perurbative knowledge of $\beta(\bar{g})$ $\rightsquigarrow$ finite-size scaling + lattice regularisation

Finite-size scaling

to bridge the gap between large scale separations

Finite-volume effects are part of the scheme / regularisation independent (e.g. lattice regulator)

- one external scale only: system size L
- fix renormalization scale appropriately

$$\mu L = \text{const}$$

- no FV corrections
- One condition: $L/a \gg 1$ (~ 10)
- phys. coupling: $\bar{g}^2 = \bar{g}^2(L)$

Finite-size scaling

to bridge the gap between large scale separations

Finite-volume effects are part of the scheme / regularisation independent (e.g. lattice regulator)

- one external scale only: system size L
- fix renormalization scale appropriately

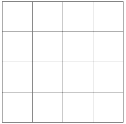
$$\mu L = \text{const}$$

- no FV corrections
- One condition: $L/a \gg 1$ (~ 10)
- phys. coupling: $\bar{g}^2 = \bar{g}^2(L)$

Step-scaling function $\Sigma(s, u, a/L)$

1. fix all bare parameter $(g_0^2, L/a)$ such that

$$\bar{g}^2(L) = u$$



$\boldsymbol{\bar{g}^2(L) \equiv u}$

$\bar{g}^2$	...
u	...

Finite-size scaling

to bridge the gap between large scale separations

Finite-volume effects are part of the scheme / regularisation independent (e.g. lattice regulator)

- one external scale only: system size L
- fix renormalization scale appropriately

$$\mu L = \text{const}$$

- no FV corrections
- One condition: $L/a \gg 1$ (~ 10)
- phys. coupling: $\bar{g}^2 = \bar{g}^2(L)$

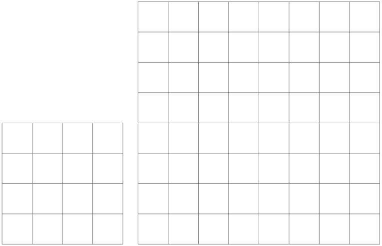
Step-scaling function $\Sigma(s, u, a/L)$

1. fix all bare parameter $(g_0^2, L/a)$ such that

$$\bar{g}^2(L) = u$$

2. measure response due to scale change

$$L \to sL \;:\; \Sigma(s, u, a/L)$$



$\bar{g}^2(L) \equiv u$ $\quad$ $\Sigma(2, u, 1/4)$

$\bar{g}^2$	$\Sigma(1/4)$	$\dots$
u	$\star$	$\dots$

Finite-size scaling

to bridge the gap between large scale separations

Finite-volume effects are part of the scheme / regularisation independent (e.g. lattice regulator)

- one external scale only: system size L
- fix renormalization scale appropriately

$$\mu L = \text{const}$$

- no FV corrections
- One condition: $L/a \gg 1$ (~ 10)
- phys. coupling: $\bar{g}^2 = \bar{g}^2(L)$

Step-scaling function $\Sigma(s, u, a/L)$

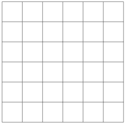
1. fix all bare parameter $(g_0^2, L/a)$ such that

$$\bar{g}^2(L) = u$$

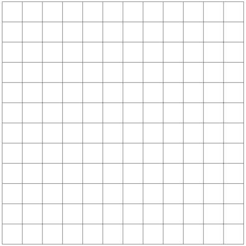
2. measure response due to scale change

$$L \to sL \ : \ \Sigma(s, u, a/L)$$

3. repeat 1+2 for various L/a



$\bar{g}^2(L) \equiv u$



$\Sigma(2, u, 1/6)$

$\bar{g}^2$	$\Sigma(1/4)$	$\Sigma(1/6)$	...
u	$\star$	$\star$	...

Finite-size scaling

to bridge the gap between large scale separations

Finite-volume effects are part of the scheme / regularisation independent (e.g. lattice regulator)

- one external scale only: system size L
- fix renormalization scale appropriately

$$\mu L = \text{const}$$

- no FV corrections
- One condition: $L/a \gg 1$ (~ 10)
- phys. coupling: $\bar{g}^2 = \bar{g}^2(L)$

Step-scaling function $\Sigma(s, u, a/L)$

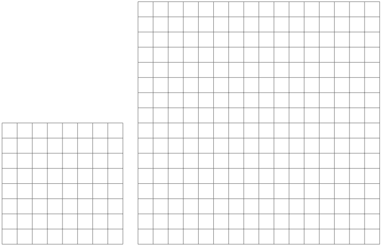
1. fix all bare parameter $(g_0^2, L/a)$ such that

$$\bar{g}^2(L) = u$$

2. measure response due to scale change

$$L \to sL \;:\; \Sigma(s, u, a/L)$$

3. repeat 1+2 for various L/a



$\bar{g}^2(L) \equiv u$ $\quad$ $\Sigma(2, u, 1/8)$

$\bar{g}^2$	$\Sigma(1/4)$	$\Sigma(1/6)$	$\Sigma(1/8)$	...
u	$\star$	$\star$	$\star$	...

Finite-size scaling

to bridge the gap between large scale separations

Finite-volume effects are part of the scheme / regularisation independent (e.g. lattice regulator)

- one external scale only: system size L
- fix renormalization scale appropriately

$$\mu L = \text{const}$$

- no FV corrections
- One condition: $L/a \gg 1$ (~ 10)
- phys. coupling: $\bar{g}^2 = \bar{g}^2(L)$

Step-scaling function $\Sigma(s, u, a/L)$

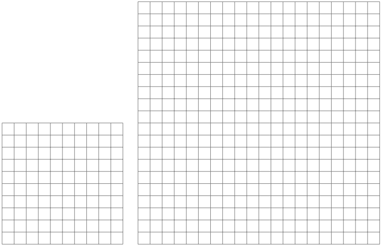
1. fix all bare parameter $(g_0^2, L/a)$ such that

$$\bar{g}^2(L) = u$$

2. measure response due to scale change

$$L \to sL \; : \; \Sigma(s, u, a/L)$$

3. repeat 1+2 for various L/a



$\bar{g}^2(L) \equiv u$ $\quad$ $\Sigma(2, u, 1/10)$

$\bar{g}^2$	$\Sigma(1/4)$	$\Sigma(1/6)$	$\Sigma(1/8)$	$\Sigma(1/10)$	...
u	$\star$	$\star$	$\star$	$\star$	...

Finite-size scaling

to bridge the gap between large scale separations

Finite-volume effects are part of the scheme / regularisation independent (e.g. lattice regulator)

- one external scale only: system size L
- fix renormalization scale appropriately

$$\mu L = \text{const}$$

- no FV corrections
- One condition: $L/a \gg 1$ (~ 10)
- phys. coupling: $\bar{g}^2 = \bar{g}^2(L)$

Step-scaling function $\Sigma(s, u, a/L)$

1. fix all bare parameter (g_0^2, L/a) such that

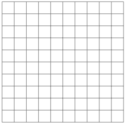
$$\bar{g}^2(L) = u$$

2. measure response due to scale change

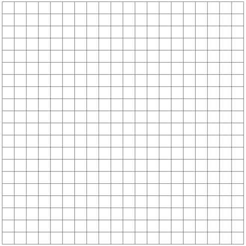
$$L \to sL \; : \; \Sigma(s, u, a/L)$$

3. repeat 1+2 for various L/a
4. continuum limit $\qquad \bar{g}^2(sL) = \sigma(s, u)$

$$\Sigma(s, u, a/L) = \sigma(s, u) + \mathcal{O}([a/L]^n)$$



$\bar{g}^2(L) \equiv u$



$\Sigma(2, u, 1/10)$

$\bar{g}^2$	$\Sigma(1/4)$	$\Sigma(1/6)$	$\Sigma(1/8)$	$\Sigma(1/10)$	...	$a \to 0$
u	$\star$	$\star$	$\star$	$\star$	...	$\sigma(u)$

Finite-size scaling

to bridge the gap between large scale separations

Finite-volume effects are part of the scheme / regularisation independent (e.g. lattice regulator)

- one external scale only: system size L
- fix renormalization scale appropriately

$$\mu L = \text{const}$$

- no FV corrections
- One condition: $L/a \gg 1$ (~ 10)
- phys. coupling: $\bar{g}^2 = \bar{g}^2(L)$

Step-scaling function $\Sigma(s, u, a/L)$

1. fix all bare parameter $(g_0^2, L/a)$ such that

$$\bar{g}^2(L) = u$$

2. measure response due to scale change

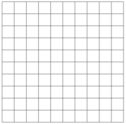
$$L \to sL \ : \ \Sigma(s, u, a/L)$$

3. repeat 1+2 for various L/a
4. continuum limit $\qquad \bar{g}^2(sL) = \sigma(s, u)$

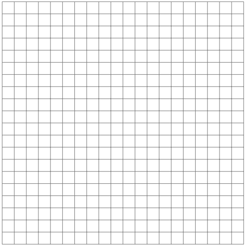
$$\Sigma(s, u, a/L) = \sigma(s, u) + \mathcal{O}([a/L]^n)$$

5. iterate 1-4 for different u k-times

$$L_{\mathrm{max}}/L_{\mathrm{min}} = s^k$$

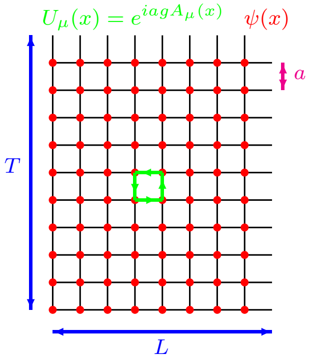


$\bar{g}^2(L) \equiv u$



$\Sigma(2, u, 1/10)$

$\bar{g}^2$	$\Sigma(1/4)$	$\Sigma(1/6)$	$\Sigma(1/8)$	$\Sigma(1/10)$	$\dots$	$a \to 0$
u	$\star$	$\star$	$\star$	$\star$	$\dots$	$\sigma(u)$
u'	$\star$	$\star$	$\star$	$\star$		$\sigma(u')$
$\dots$						



- Path integral representation

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[U, \overline{\psi}, \psi]\, O(U, \overline{\psi}, \psi)\, \mathrm{e}^{-S_\mathrm{G}[U] - S_\mathrm{F}[U, \overline{\psi}, \psi]}$$

$$= \frac{1}{\mathcal{Z}} \int \mathcal{D}[U]\, [O(U)]_{\text{Wick}}\, \mathrm{e}^{-S_\mathrm{G}[U]} \det(D)$$

- Evaluate integral numerically through *Monte Carlo sampling* of $\mathrm{e}^{-S_\mathrm{G}[U]} \det(D) \geq 0$.
- Observable computed by averaging over samples

$$\langle O \rangle = \frac{1}{N_{\text{cfg}}} \sum_{i=1}^{N_{\text{cfg}}} O(U_i) + \mathcal{O}(1/\sqrt{N_{\text{cfg}}})$$

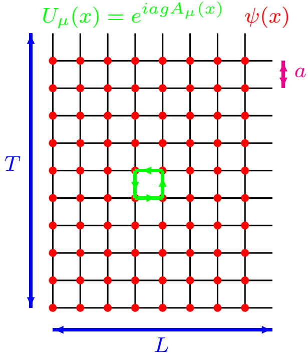
including a *careful statistical error analysis!*

Gauge action as sum over all plaquettes p:

$$S_\mathrm{G}[U] = \frac{\beta}{2N} \sum_p \mathrm{Tr}(1 - U_p - U_p^\dagger) \xrightarrow[a \to 0]{} -\frac{1}{2} \int \mathrm{d}^4 x\, \mathrm{Tr}(F_{\mu\nu} F_{\mu\nu})\,, \qquad a = a(\beta)$$

For one simulation choose: $(T/L, L/a, \beta, \{\kappa_i\})$ + boundary conditions of the fields

Lattice methods in a jiffy



- Path integral representation

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[U,\overline{\psi},\psi]\, O(U,\overline{\psi},\psi)\, \mathrm{e}^{-S_{\rm G}[U]-S_{\rm F}[U,\overline{\psi},\psi]}$$
$$= \frac{1}{\mathcal{Z}} \int \mathcal{D}[U]\, [O(U)]_{\text{Wick}}\, \mathrm{e}^{-S_{\rm G}[U]} \det(D)$$

- Evaluate integral numerically through *Monte Carlo sampling* of $\mathrm{e}^{-S_{\rm G}[U]} \det(D) \geq 0$.
- Observable computed by averaging over samples

$$\langle O \rangle = \frac{1}{N_{\text{cfg}}} \sum_{i=1}^{N_{\text{cfg}}} O(U_i) + \mathcal{O}(1/\sqrt{N_{\text{cfg}}})$$

including a *careful statistical error analysis!*

Gauge action as sum over all plaquettes p:

$$S_{\rm G}[U] = \frac{\beta}{2N} \sum_p \mathrm{Tr}(1 - U_p - U_p^\dagger) \xrightarrow[a\to 0]{} -\frac{1}{2} \int \mathrm{d}^4x\, \mathrm{Tr}(F_{\mu\nu}F_{\mu\nu})\,, \qquad a = a(\beta)$$

For one simulation choose: $(T/L, L/a, \beta, \{\kappa_i\})$ + boundary conditions of the fields

Q: But what observable $O(\mu)$ do we use to define the renormalised coupling?

The gradient flow

flow $B_\mu(x,t)$ of $SU(N)$ gauge fields driven by $(t \geq 0)$: $x = (x_0, \mathbf{x})$

$$\frac{\mathrm{d}B_\mu(x,t)}{\mathrm{d}t} = D_\nu G_{\nu\mu}(x,t) \,, \qquad \left(\propto -\frac{\delta S_{\mathrm{YM}}[B]}{\delta B_\mu}\right)$$

$$B_\mu(x,t)|_{t=0} = A_\mu(x) \,,$$: initial condition

$$G_{\mu\nu}(x,t) = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] \,, \quad D_\nu = \partial_\nu + [B_\nu, \star]$$

The gradient flow

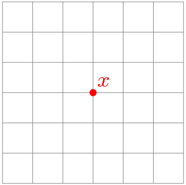
flow $B_\mu(x,t)$ of $SU(N)$ gauge fields driven by $(t \geq 0)$:

$x = (x_0, \mathbf{x})$

$$\frac{\mathrm{d}B_\mu(x,t)}{\mathrm{d}t} = D_\nu G_{\nu\mu}(x,t)\,, \qquad \left(\propto -\frac{\delta S_{\mathrm{YM}}[B]}{\delta B_\mu}\right)$$

$$B_\mu(x,t)|_{t=0} = A_\mu(x)\,, \qquad \text{: initial condition}$$

$$G_{\mu\nu}(x,t) = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]\,, \qquad D_\nu = \partial_\nu + [B_\nu, \star]$$



$t = 0$

Properties:

- continuous smoothing of gauge fields with flow time t [Lüscher:'10]
  $\Rightarrow$ UV fluctuations get suppressed as B_μ flows towards classical solution

The gradient flow

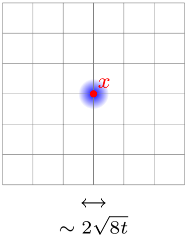
flow $B_\mu(x,t)$ of $SU(N)$ gauge fields driven by $(t \geq 0)$:

$x = (x_0, \mathbf{x})$

$$\frac{\mathrm{d}B_\mu(x,t)}{\mathrm{d}t} = D_\nu G_{\nu\mu}(x,t) \,, \qquad \left(\propto -\frac{\delta S_{\mathrm{YM}}[B]}{\delta B_\mu}\right)$$

$$B_\mu(x,t)|_{t=0} = A_\mu(x) \,,$$: initial condition

$$G_{\mu\nu}(x,t) = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] \,, \quad D_\nu = \partial_\nu + [B_\nu, \star]$$



Properties:

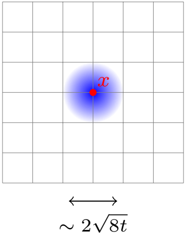
- continuous smoothing of gauge fields with flow time t [Lüscher:'10]
  $\Rightarrow$ UV fluctuations get suppressed as B_μ flows towards classical solution

flow $B_\mu(x,t)$ of $SU(N)$ gauge fields driven by $(t \geq 0)$: $x = (x_0, \mathbf{x})$

$$\frac{\mathrm{d}B_\mu(x,t)}{\mathrm{d}t} = D_\nu G_{\nu\mu}(x,t)\,, \qquad \left(\propto -\frac{\delta S_{\mathrm{YM}}[B]}{\delta B_\mu}\right)$$

$$B_\mu(x,t)|_{t=0} = A_\mu(x)\,,$$: initial condition

$$G_{\mu\nu}(x,t) = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]\,, \quad D_\nu = \partial_\nu + [B_\nu, \star]$$



Properties:

- continuous smoothing of gauge fields with flow time t [Lüscher:'10]
  $\Rightarrow$ UV fluctuations get suppressed as B_μ flows towards classical solution

The gradient flow

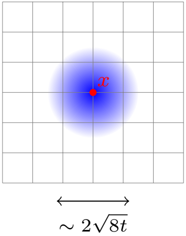
flow $B_\mu(x,t)$ of $SU(N)$ gauge fields driven by $(t \geq 0)$:

$x = (x_0, \mathbf{x})$

$$\frac{\mathrm{d}B_\mu(x,t)}{\mathrm{d}t} = D_\nu G_{\nu\mu}(x,t)\,, \qquad \left(\propto -\frac{\delta S_{\mathrm{YM}}[B]}{\delta B_\mu}\right)$$

$$B_\mu(x,t)|_{t=0} = A_\mu(x)\,,$$: initial condition

$$G_{\mu\nu}(x,t) = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]\,, \quad D_\nu = \partial_\nu + [B_\nu, \star]$$



Properties:

- continuous smoothing of gauge fields with flow time t [Lüscher:'10]
  $\Rightarrow$ UV fluctuations get suppressed as B_μ flows towards classical solution

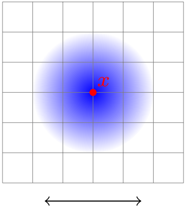
flow $B_\mu(x,t)$ of $SU(N)$ gauge fields driven by $(t \geq 0)$:

$x = (x_0, \mathbf{x})$

$$\frac{\mathrm{d}B_\mu(x,t)}{\mathrm{d}t} = D_\nu G_{\nu\mu}(x,t)\,, \qquad \left(\propto -\frac{\delta S_{\mathrm{YM}}[B]}{\delta B_\mu}\right)$$

$$B_\mu(x,t)|_{t=0} = A_\mu(x)\,, \qquad \text{: initial condition}$$

$$G_{\mu\nu}(x,t) = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]\,, \quad D_\nu = \partial_\nu + [B_\nu, \star]$$



$\sim 2\sqrt{8t}$

Properties:

- continuous smoothing of gauge fields with flow time t [Lüscher:'10]
  $\Rightarrow$ UV fluctuations get suppressed as B_μ flows towards classical solution

The gradient flow

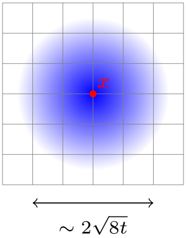
flow $B_\mu(x,t)$ of $SU(N)$ gauge fields driven by $(t \geq 0)$:

$x = (x_0, \mathbf{x})$

$$\frac{\mathrm{d}B_\mu(x,t)}{\mathrm{d}t} = D_\nu G_{\nu\mu}(x,t)\,, \qquad \left(\propto -\frac{\delta S_{\mathrm{YM}}[B]}{\delta B_\mu}\right)$$

$$B_\mu(x,t)|_{t=0} = A_\mu(x)\,, \qquad \text{: initial condition}$$

$$G_{\mu\nu}(x,t) = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]\,, \quad D_\nu = \partial_\nu + [B_\nu, \star]$$



Properties:

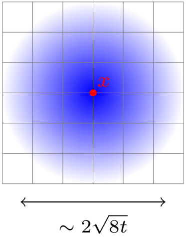
- continuous smoothing of gauge fields with flow time t [Lüscher:'10]
  $\Rightarrow$ UV fluctuations get suppressed as B_μ flows towards classical solution

flow $B_\mu(x,t)$ of $SU(N)$ gauge fields driven by $(t \geq 0)$: $x = (x_0, \mathbf{x})$

$$\frac{\mathrm{d}B_\mu(x,t)}{\mathrm{d}t} = D_\nu G_{\nu\mu}(x,t)\,, \qquad \left(\propto -\frac{\delta S_{\rm YM}[B]}{\delta B_\mu}\right)$$

$$B_\mu(x,t)|_{t=0} = A_\mu(x)\,, \qquad \text{: initial condition}$$

$$G_{\mu\nu}(x,t) = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]\,, \quad D_\nu = \partial_\nu + [B_\nu, \star]$$



Properties:

- continuous smoothing of gauge fields with flow time t [Lüscher:'10]
  $\Rightarrow$ UV fluctuations get suppressed as B_μ flows towards classical solution

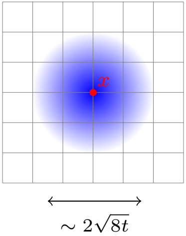
The gradient flow

flow $B_\mu(x,t)$ of $SU(N)$ gauge fields driven by $(t \geq 0)$: $x = (x_0, \mathbf{x})$

$$\frac{\mathrm{d}B_\mu(x,t)}{\mathrm{d}t} = D_\nu G_{\nu\mu}(x,t)\,, \qquad \left(\propto -\frac{\delta S_{\mathrm{YM}}[B]}{\delta B_\mu}\right)$$

$$B_\mu(x,t)|_{t=0} = A_\mu(x)\,,$$: initial condition

$$G_{\mu\nu}(x,t) = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]\,, \quad D_\nu = \partial_\nu + [B_\nu, \star]$$



Properties:

- continuous smoothing of gauge fields with flow time t [Lüscher:'10]
  $\Rightarrow$ UV fluctuations get suppressed as B_μ flows towards classical solution
- UV finite at $t > 0$, *proven to all orders in PT for YM theory* [Lüscher,Weisz:'11]
  $\Rightarrow$ ∃CL for correlation functions $G(x_1, x_2, \ldots) = \langle B(x_1,t) B(x_2,t) \cdots \rangle$

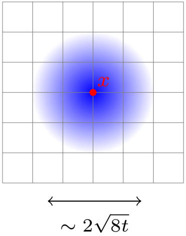
The gradient flow

flow $B_\mu(x,t)$ of $SU(N)$ gauge fields driven by $(t \geq 0)$: $x = (x_0, \mathbf{x})$

$$\frac{\mathrm{d}B_\mu(x,t)}{\mathrm{d}t} = D_\nu G_{\nu\mu}(x,t)\,, \qquad \left(\propto -\frac{\delta S_{\mathrm{YM}}[B]}{\delta B_\mu}\right)$$

$$B_\mu(x,t)|_{t=0} = A_\mu(x)\,,$$: initial condition

$$G_{\mu\nu}(x,t) = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]\,, \qquad D_\nu = \partial_\nu + [B_\nu, \star]$$



Properties:

- continuous smoothing of gauge fields with flow time t [Lüscher:'10]
  $\Rightarrow$ UV fluctuations get suppressed as B_μ flows towards classical solution
- UV finite at $t > 0$, *proven to all orders in PT for YM theory* [Lüscher,Weisz:'11]
  $\Rightarrow$ ∃CL for correlation functions $G(x_1, x_2, \ldots) = \langle B(x_1,t)B(x_2,t)\cdots\rangle$
- excellent numerical precision, cheap, …

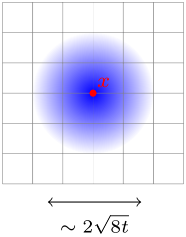
The gradient flow

flow $B_\mu(x,t)$ of $SU(N)$ gauge fields driven by $(t \geq 0)$: $x = (x_0, \mathbf{x})$

$$\frac{\mathrm{d}B_\mu(x,t)}{\mathrm{d}t} = D_\nu G_{\nu\mu}(x,t)\,, \qquad \left(\propto -\frac{\delta S_{\mathrm{YM}}[B]}{\delta B_\mu}\right)$$

$$B_\mu(x,t)|_{t=0} = A_\mu(x)\,,$$: initial condition

$$G_{\mu\nu}(x,t) = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]\,, \qquad D_\nu = \partial_\nu + [B_\nu, \star]$$



Properties:

- continuous smoothing of gauge fields with flow time t [Lüscher:'10]
  $\Rightarrow$ UV fluctuations get suppressed as B_μ flows towards classical solution
- UV finite at $t > 0$, *proven to all orders in PT for YM theory* [Lüscher,Weisz:'11]
  $\Rightarrow$ $\exists$CL for correlation functions $G(x_1, x_2, \ldots) = \langle B(x_1,t) B(x_2,t) \cdots \rangle$
- excellent numerical precision, cheap, …

E.g.: Energy density in pure YM

$$\langle E(x,t) \rangle = \tfrac{1}{4} \langle G^a_{\mu\nu}(x,t) G^a_{\mu\nu}(x,t) \rangle$$

is finite (for $t > 0$) after the usual coupling renormalization.

The gradient flow coupling

[Lüscher:'10]

for $SU(N)$, $D=4$, $V=\infty$

$$\langle E(t)\rangle \equiv \frac{1}{4}\left\langle G^a_{\mu\nu}(t)G^a_{\mu\nu}(t)\right\rangle = \frac{3(N^2-1)}{2(8\pi t)^2}\times \bar{g}^2_{\rm MS}(\mu)\left\{1+c_2\bar{g}^2_{\rm MS}+{\rm O}\left(\bar{g}^4_{\rm MS}\right)\right\}$$

and $\mu = 1/\sqrt{8t}$

- $r=\sqrt{8t}$ is effective smoothing radius of Wilson flow
- automatic renormalized at $t>0$

The gradient flow coupling

[Lüscher:'10]

for $SU(N)$, $D=4$, $V=\infty$

$$\langle E(t)\rangle \equiv \frac{1}{4}\left\langle G^a_{\mu\nu}(t)G^a_{\mu\nu}(t)\right\rangle = \frac{3(N^2-1)}{2(8\pi t)^2} \times \bar{g}^2_{\rm MS}(\mu)\Big\{1 + c_2\bar{g}^2_{\rm MS} + \mathrm{O}\big(\bar{g}^4_{\rm MS}\big)\Big\}$$

and $\mu = 1/\sqrt{8t}$

- $r=\sqrt{8t}$ is effective smoothing radius of Wilson flow
- automatic renormalized at $t>0$
- provides non-perturbative definition of a 'gradient flow' coupling in finite-volume $L^3\times T$

$$t^2\langle E(t)\rangle = \mathcal{N}\cdot\bar{g}^2(\mu)\;, \qquad \mu^{-1}=\sqrt{8t}=cL\;, \qquad \forall N, N_{\rm f}$$

  - $\mathcal{N}$ normalization factor such that $\bar{g}^2 = g_0^2 + \mathrm{O}\big(g_0^4\big)$
  - $c=\sqrt{8t}/L$ effective smoothing range of Wilson flow

The gradient flow coupling

[Lüscher:'10]

for $SU(N)$, $D = 4$, $V = \infty$

$$\langle E(t)\rangle \equiv \frac{1}{4}\left\langle G^a_{\mu\nu}(t) G^a_{\mu\nu}(t)\right\rangle = \frac{3(N^2-1)}{2(8\pi t)^2} \times \bar{g}^2_{\rm MS}(\mu)\left\{1 + c_2 \bar{g}^2_{\rm MS} + \mathrm{O}\left(\bar{g}^4_{\rm MS}\right)\right\}$$

and $\mu = 1/\sqrt{8t}$

- $r = \sqrt{8t}$ is effective smoothing radius of Wilson flow
- automatic renormalized at $t > 0$
- provides non-perturbative definition of a 'gradient flow' coupling in finite-volume $L^3 \times T$

$$t^2\langle E(t)\rangle = \mathcal{N} \cdot \bar{g}^2(\mu)\ , \qquad \mu^{-1} = \sqrt{8t} = cL\ , \qquad \forall N, N_{\rm f}$$

  - $\mathcal{N}$ normalization factor such that $\bar{g}^2 = g_0^2 + \mathrm{O}\left(g_0^4\right)$
  - $c = \sqrt{8t}/L$ effective smoothing range of Wilson flow
  - in finite volume boundary conditions become important:
    - $\mathcal{N}$ for periodic boundary conditions [Fodor et al.'12]
    - $\mathcal{N}$ for Schrödinger functional boundary conditions [Fritzsch,Ramos:1301.4388]
    - $\mathcal{N}$ for twisted boundary conditions [Ramos:PosLattice'13]

Normalization factor $\mathcal{N}$

computed with SF boundary conditions

[PF,Ramos:JHEP1310(2013)008]

rescaling & expanding the gauge fields at $t > 0$

$$B_\mu = \sum_{n=1}^{\infty} B_{\mu,n} g_0^n \ , \qquad \langle E(t,x_0)\rangle = \sum_{n=0}^{\infty} \mathcal{E}_n(t,x_0) \ , \text{ with } \quad \mathcal{E}_n = \mathrm{O}\big(g_0^{2+n}\big)$$

LO: $\mathcal{E}_0(t,x_0) = \frac{g_0^2}{2}\langle \partial_\mu B^a_{\nu,1}\partial_\mu B^a_{\nu,1} - \partial_\mu B^a_{\nu,1}\partial_\nu B^a_{\mu,1}\rangle$

expanding flow equation $\longrightarrow$ tower of equations

Normalization factor $\mathcal{N}$

computed with SF boundary conditions

[PF,Ramos:JHEP1310(2013)008]

rescaling & expanding the gauge fields at $t > 0$

$$B_\mu = \sum_{n=1}^{\infty} B_{\mu,n} g_0^n \,, \qquad \langle E(t,x_0)\rangle = \sum_{n=0}^{\infty} \mathcal{E}_n(t,x_0) \,, \text{ with } \quad \mathcal{E}_n = \mathrm{O}\big(g_0^{2+n}\big)$$

LO: $\mathcal{E}_0(t,x_0) = \frac{g_0^2}{2}\langle \partial_\mu B^a_{\nu,1}\partial_\mu B^a_{\nu,1} - \partial_\mu B^a_{\nu,1}\partial_\nu B^a_{\mu,1}\rangle$

expanding flow equation $\longrightarrow$ tower of equations

LO: Flow equation = Heat equation $\quad \dot{B}_{\mu,1} = \partial_\nu\partial_\nu B_{\mu,1}$ + gauge terms

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{B}_{\mu,1}(\mathbf{p},x_0,t) = (-\mathbf{p}^2 + \partial_0^2)\tilde{B}_{\mu,1}(\mathbf{p},x_0,t) \,, \qquad \tilde{B}_{\mu,1}(\mathbf{p},x_0,t)\big|_{t=0} = \tilde{A}_\mu(\mathbf{p},x_0)$$

- heat kernels need to comply with boundary conditions $\rightarrow K^D, K^N, K^{ND}, \ldots$

Normalization factor $\mathcal{N}$

computed with SF boundary conditions

rescaling & expanding the gauge fields at $t > 0$

$$B_\mu = \sum_{n=1}^{\infty} B_{\mu,n} g_0^n \,, \qquad \langle E(t,x_0)\rangle = \sum_{n=0}^{\infty} \mathcal{E}_n(t,x_0)\,, \text{ with } \quad \mathcal{E}_n = \mathrm{O}\big(g_0^{2+n}\big)$$

LO: $$\mathcal{E}_0(t,x_0) = \frac{g_0^2}{2}\langle \partial_\mu B^a_{\nu,1}\partial_\mu B^a_{\nu,1} - \partial_\mu B^a_{\nu,1}\partial_\nu B^a_{\mu,1}\rangle$$

expanding flow equation $\longrightarrow$ tower of equations

LO: Flow equation = Heat equation $\qquad \dot{B}_{\mu,1} = \partial_\nu\partial_\nu B_{\mu,1}$ + gauge terms

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{B}_{\mu,1}(\mathbf{p},x_0,t) = (-\mathbf{p}^2 + \partial_0^2)\tilde{B}_{\mu,1}(\mathbf{p},x_0,t)\,, \qquad \tilde{B}_{\mu,1}(\mathbf{p},x_0,t)\big|_{t=0} = \tilde{A}_\mu(\mathbf{p},x_0)$$

- heat kernels need to comply with boundary conditions $\rightarrow K^D, K^N, K^{ND}, \ldots$

$$B_{\mu,1}(\mathbf{x},x_0,t) = \int \mathrm{d}\mathbf{x}' \int \mathrm{d}x_0' \prod_{i=1}^{3} K^P(x_i,x_i',t) K^{D,N}(x_0,x_0',t) A_\mu(\mathbf{x}',x_0',t)$$

$\downarrow$ LO

$$\langle E(t)\rangle = \mathcal{E}_0(t) = \partial\partial \int [\text{heat kernel stuff}]\langle AA\rangle \qquad \Rightarrow \mathcal{N}(c, T/L, x_0/T)$$

Normalization factor $\mathcal{N}$

computed with SF boundary conditions

rescaling & expanding the gauge fields at $t > 0$

$$B_\mu = \sum_{n=1}^{\infty} B_{\mu,n} g_0^n \,, \qquad \langle E(t,x_0)\rangle = \sum_{n=0}^{\infty} \mathcal{E}_n(t,x_0) \,, \text{ with } \quad \mathcal{E}_n = \mathrm{O}\big(g_0^{2+n}\big)$$

LO: $\mathcal{E}_0(t,x_0) = \frac{g_0^2}{2}\langle \partial_\mu B^a_{\nu,1}\partial_\mu B^a_{\nu,1} - \partial_\mu B^a_{\nu,1}\partial_\nu B^a_{\mu,1}\rangle$

expanding flow equation $\longrightarrow$ tower of equations

LO: Flow equation = Heat equation $\quad \dot{B}_{\mu,1} = \partial_\nu\partial_\nu B_{\mu,1}$ + gauge terms

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{B}_{\mu,1}(\mathbf{p},x_0,t) = (-\mathbf{p}^2 + \partial_0^2)\tilde{B}_{\mu,1}(\mathbf{p},x_0,t) \,, \qquad \tilde{B}_{\mu,1}(\mathbf{p},x_0,t)\big|_{t=0} = \tilde{A}_\mu(\mathbf{p},x_0)$$

- heat kernels need to comply with boundary conditions $\rightarrow K^D, K^N, K^{ND}, \ldots$

$$B_{\mu,1}(\mathbf{x},x_0,t) = \int \mathrm{d}\mathbf{x}' \int \mathrm{d}x_0' \prod_{i=1}^{3} K^P(x_i,x_i',t) K^{D,N}(x_0,x_0',t) A_\mu(\mathbf{x}',x_0',t)$$

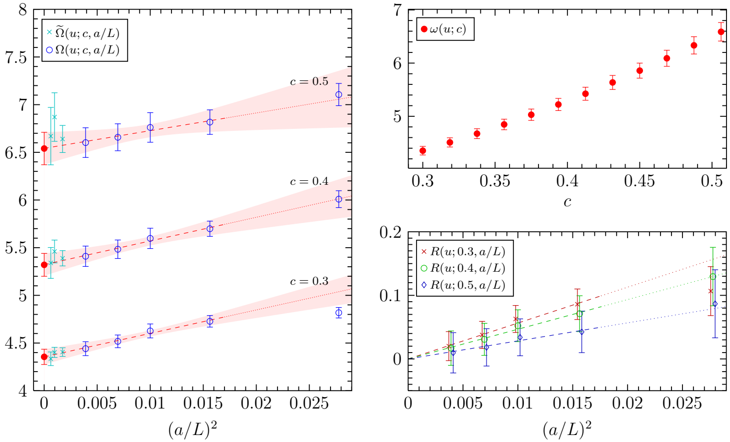
$\downarrow$ LO

$$\langle E(t)\rangle = \mathcal{E}_0(t) = \partial\partial \int [\text{heat kernel stuff}]\langle AA\rangle \qquad \Rightarrow \mathcal{N}(c, T/L, x_0/T)$$

computation of lattice norm along the same lines: $\qquad \Rightarrow \widehat{\mathcal{N}}(c, T/L, x_0/T, a/L)$

SF with plaq. action and flow & clover def. of $E(t, x_0)$

$\bar{g}^2_{\mathrm{SF}}(L)$ fixed to $u = 4.484$, $L \approx 0.4\,\mathrm{fm}$ $(N_\mathrm{f} = 2, m_\mathrm{sea} = 0, \theta = 0.5$,BF=0)

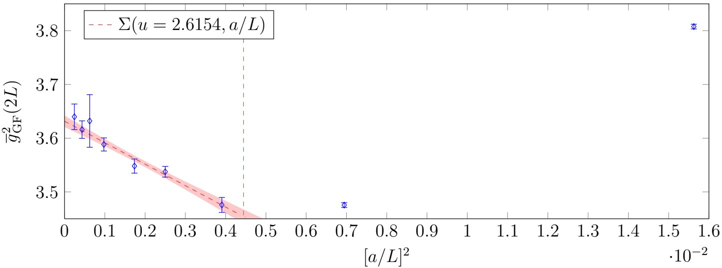


$$\bar{g}^2_{\mathrm{GF}}(L) = \lim_{a/L \to 0} \Omega(u, c, a/L)$$

Test: SSF in pure YM

SF with TLI LW gauge action, plaq. flow & clover def. of $E(t, x_0)$

compute SSF: $\sigma(u) = \lim_{a/L \to 0} \Sigma(s, u, a/L)$ $s = 2,\ L/a = 4, 6, 8, 10, 12, 16, 20, 24$



General properties of GF coupling:

- very easy to compute, negligible computational costs
- rather large cutoff effects — can & will be improved
- optimal window: $0.25 \le c = \sqrt{8t}/L \le 0.5$ — cutoff effects vs. statistical error
- scaling for $L/a > 8$

So far, so good

What does that mean for our goal to compute Λ more accurately?

So far, so good

What does that mean for our goal to compute Λ more accurately?

Renormalization group: $\beta(\bar{g}) \overset{\bar{g}\to 0}{\sim} -\bar{g}^3\left(b_0 + b_1\bar{g}^2 + \ldots\right)$

with each step one accumulates an error: $\frac{\Delta L}{L} = -\frac{\Delta\bar{g}^2}{2\bar{g}\beta(\bar{g})} \sim \frac{\Delta\bar{g}^2}{2b_0\bar{g}^4} \sim \frac{1}{\bar{g}^2}\left[\frac{\Delta\bar{g}^2}{\bar{g}^2}\right]$

Relative variance of O

$$\mathcal{V}_O = \frac{\langle O^2\rangle - \langle O\rangle^2}{\langle O\rangle^2}$$

So far, so good

What does that mean for our goal to compute Λ more accurately?

Renormalization group: $$\beta(\bar{g}) \overset{\bar{g}\to 0}{\sim} -\bar{g}^3 \left(b_0 + b_1 \bar{g}^2 + \ldots\right)$$

with each step one accumulates an error: $$\frac{\Delta L}{L} = -\frac{\Delta \bar{g}^2}{2\bar{g}\beta(\bar{g})} \sim \frac{\Delta \bar{g}^2}{2 b_0 \bar{g}^4} \sim \frac{1}{\bar{g}^2}\left[\frac{\Delta \bar{g}^2}{\bar{g}^2}\right]$$

Relative variance of O

$$\mathcal{V}_O = \frac{\langle O^2\rangle - \langle O\rangle^2}{\langle O\rangle^2}$$

Gradient flow coupling

$$\frac{\Delta \bar{g}^2_{\mathrm{GF}}}{\bar{g}^2_{\mathrm{GF}}} \sim \text{const} \quad \Rightarrow \quad \frac{\Delta L}{L} \sim \frac{\text{const}}{\bar{g}^2_{\mathrm{GF}}}$$

✓ @ large $\bar{g}^2_{\mathrm{GF}}$, × @ small $\bar{g}^2_{\mathrm{GF}}$

- Bad for $\Delta\Lambda$ with 10 finite-size scaling steps!
- more expensive due to larger L/a

So far, so good

What does that mean for our goal to compute Λ more accurately?

Renormalization group: $$\beta(\bar{g}) \overset{\bar{g}\to 0}{\sim} -\bar{g}^3 \left(b_0 + b_1 \bar{g}^2 + \ldots\right)$$

with each step one accumulates an error: $$\frac{\Delta L}{L} = -\frac{\Delta \bar{g}^2}{2\bar{g}\beta(\bar{g})} \sim \frac{\Delta \bar{g}^2}{2 b_0 \bar{g}^4} \sim \frac{1}{\bar{g}^2}\left[\frac{\Delta \bar{g}^2}{\bar{g}^2}\right]$$

Relative variance of O

$$\mathcal{V}_O = \frac{\langle O^2\rangle - \langle O\rangle^2}{\langle O\rangle^2}$$

Gradient flow coupling

$$\frac{\Delta \bar{g}^2_{\mathrm{GF}}}{\bar{g}^2_{\mathrm{GF}}} \sim \text{const} \quad \Rightarrow \quad \frac{\Delta L}{L} \sim \frac{\text{const}}{\bar{g}^2_{\mathrm{GF}}}$$

✓ @ large $\bar{g}^2_{\mathrm{GF}}$, × @ small $\bar{g}^2_{\mathrm{GF}}$

- Bad for $\Delta\Lambda$ with 10 finite-size scaling steps!
- more expensive due to larger L/a

there is a solution

$$\frac{\Delta \bar{g}^2}{\bar{g}^2} \sim \bar{g}^2 \quad \Rightarrow \quad \frac{\Delta L}{L} \sim \text{const}$$

× @ large $\bar{g}^2$, ✓ @ small $\bar{g}^2$

The Schrödinger functional coupling

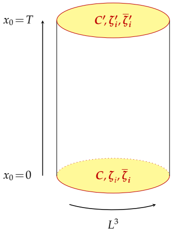
[LüscherEtAl:'92,...]

- Euclidean partition function

$$\mathcal{Z} \equiv \int_{T\times L^3} \mathcal{D}[U,\overline{\psi},\psi]\, \mathrm{e}^{-S[U,\overline{\psi},\psi]} = \langle 0|\, \mathrm{e}^{-TH} P|0\rangle$$

  with *periodic BC in L^3*
  and *Dirichlet BC in T* (breaking translational inv. in time)

- renormalization scale $\mu \propto L^{-1}$ (for step-scaling)
- mass-independent scheme, ...



Abelian boundary fields: $C_k = \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{pmatrix}$; $C'_k = \begin{pmatrix} \phi'_1 & 0 & 0 \\ 0 & \phi'_2 & 0 \\ 0 & 0 & \phi'_3 \end{pmatrix}$

SF coupling

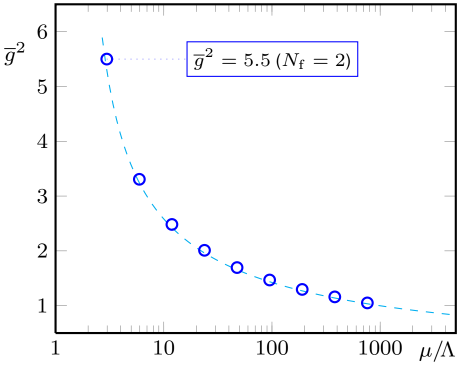
defined as variation of effective action $\Gamma = -\ln \mathcal{Z}[C,C']$,

$$\left.\frac{\partial\Gamma}{\partial\eta}\right|_{\eta=0} = \frac{\mathrm{const}}{\overline{g}^2_{\mathrm{SF}}(L)}$$

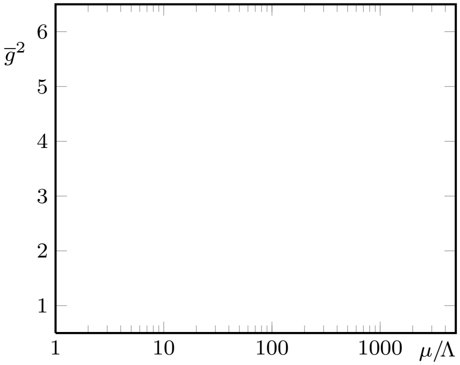
for non-vanishing boundary gauge fields $C_k \neq 0 \neq C'_k$

Our new strategy

Combine SF and GF running couplings



Combine SF and GF running couplings

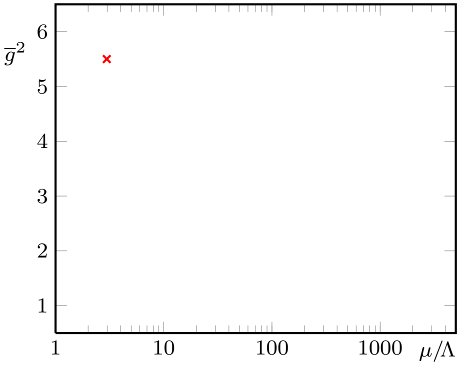


1. physical scale $L_{\max}$ from LV (CLS) runs
   [LW action, $m_\pi \neq 0$, $m_{\rm K} \neq 0$]

   $$\bar{g}^2_{\rm GF}(L_{\max}, c, \varphi)$$

   (maybe massive scheme?)

Combine SF and GF running couplings



1. physical scale $L_{\max}$ from LV (CLS) runs
   [LW action, $m_\pi \neq 0$, $m_{\mathrm{K}} \neq 0$]

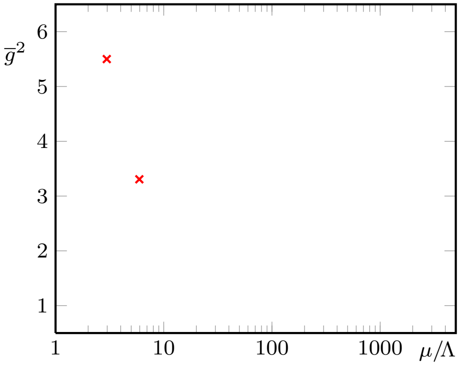
   $$\bar{g}^2_{\mathrm{GF}}(L_{\max}, c, \varphi)$$

   (maybe massive scheme?)

2. step scaling $L_{\max} \to L_{\max}/2 \to \ldots$
   (change to massless scheme?)

   $$\bar{g}^2_{\mathrm{GF}}(L_{\max}/4, c, 0)$$

Combine SF and GF running couplings

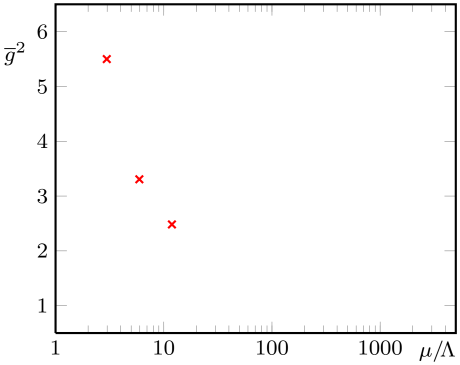


1. physical scale $L_{\max}$ from LV (CLS) runs
   [LW action, $m_\pi \neq 0$, $m_\mathrm{K} \neq 0$]

   $$\bar{g}^2_\mathrm{GF}(L_{\max}, c, \varphi)$$

   (maybe massive scheme?)
2. step scaling $L_{\max} \to L_{\max}/2 \to \dots$
   (change to massless scheme?)

   $$\bar{g}^2_\mathrm{GF}(L_{\max}/4, c, 0)$$
3. step scaling $L_{\max}/4 \to L_{\max}/8 \to \dots$



1. physical scale $L_{\max}$ from LV (CLS) runs
   [LW action, $m_\pi \neq 0$, $m_K \neq 0$]

   $$\bar{g}^2_{\rm GF}(L_{\max}, c, \varphi)$$

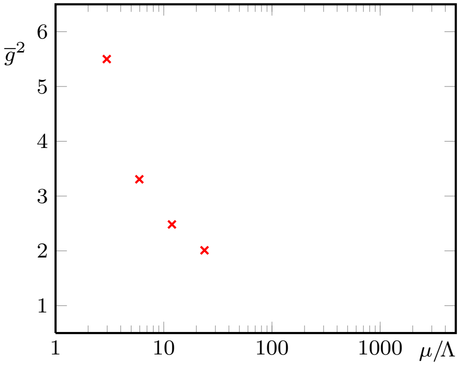
   (maybe massive scheme?)

2. step scaling $L_{\max} \to L_{\max}/2 \to \ldots$
   (change to massless scheme?)

   $$\bar{g}^2_{\rm GF}(L_{\max}/4, c, 0)$$

3. step scaling $L_{\max}/4 \to L_{\max}/8 \to \ldots$

Combine SF and GF running couplings



1. physical scale $L_{\rm max}$ from LV (CLS) runs
   [LW action, $m_\pi \neq 0$, $m_{\rm K} \neq 0$]

   $$\bar{g}^2_{\rm GF}(L_{\rm max}, c, \varphi)$$

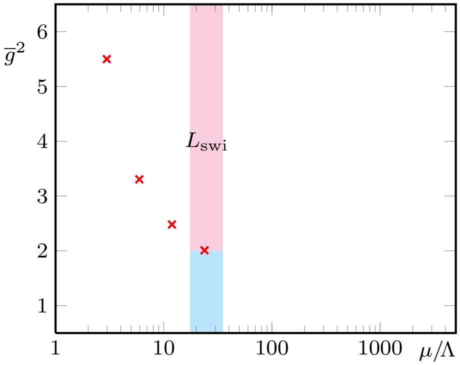
   (maybe massive scheme?)

2. step scaling $L_{\rm max} \to L_{\rm max}/2 \to \ldots$
   (change to massless scheme?)

   $$\bar{g}^2_{\rm GF}(L_{\rm max}/4, c, 0)$$

3. step scaling $L_{\rm max}/4 \to L_{\rm max}/8 \to \ldots$

Combine SF and GF running couplings



1. physical scale $L_{\rm max}$ from LV (CLS) runs
   [LW action, $m_\pi \neq 0$, $m_{\rm K} \neq 0$]

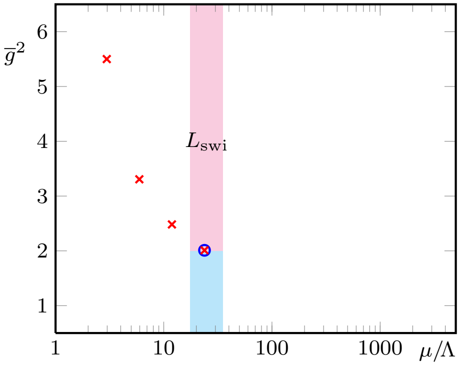
   $$\bar{g}^2_{\rm GF}(L_{\rm max}, c, \varphi)$$

   (maybe massive scheme?)
2. step scaling $L_{\rm max} \to L_{\rm max}/2 \to \ldots$
   (change to massless scheme?)

   $$\bar{g}^2_{\rm GF}(L_{\rm max}/4, c, 0)$$

3. step scaling $L_{\rm max}/4 \to L_{\rm max}/8 \to \ldots$
4. precision alert at some scale $L_{\rm switch}$

Combine SF and GF running couplings



1. physical scale $L_{\rm max}$ from LV (CLS) runs
   [LW action, $m_\pi \neq 0$, $m_{\rm K} \neq 0$]

   $$\bar{g}^2_{\rm GF}(L_{\rm max}, c, \varphi)$$

   (maybe massive scheme?)

2. step scaling $L_{\rm max} \to L_{\rm max}/2 \to \ldots$
   (change to massless scheme?)

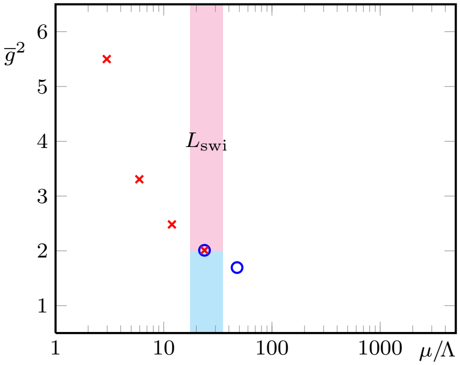
   $$\bar{g}^2_{\rm GF}(L_{\rm max}/4, c, 0)$$

3. step scaling $L_{\rm max}/4 \to L_{\rm max}/8 \to \ldots$
4. precision alert at some scale $L_{\rm switch}$
5. match $\bar{g}^2_{\rm SF}$ & $\bar{g}^2_{\rm GF}$ NP'ly, fix $u = \bar{g}^2_{\rm GF}(L_{\rm swi})$

   $$\bar{g}^2_{\rm SF}(L_{\rm swi}) = \lim_{a/L \to 0} \Psi(u, a/L)$$

   or vice versa

Combine SF and GF running couplings



1. physical scale $L_{\rm max}$ from LV (CLS) runs
   [LW action, $m_\pi \neq 0$, $m_{\rm K} \neq 0$]

   $$\bar{g}^2_{\rm GF}(L_{\rm max}, c, \varphi)$$

   (maybe massive scheme?)
2. step scaling $L_{\rm max} \to L_{\rm max}/2 \to \ldots$
   (change to massless scheme?)

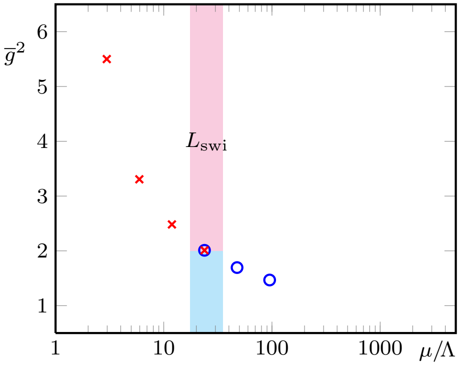
   $$\bar{g}^2_{\rm GF}(L_{\rm max}/4, c, 0)$$

3. step scaling $L_{\rm max}/4 \to L_{\rm max}/8 \to \ldots$
4. precision alert at some scale $L_{\rm switch}$
5. match $\bar{g}^2_{\rm SF}$ & $\bar{g}^2_{\rm GF}$ NP'ly, fix $u = \bar{g}^2_{\rm GF}(L_{\rm swi})$

   $$\bar{g}^2_{\rm SF}(L_{\rm swi}) = \lim_{a/L\to 0} \Psi(u, a/L)$$

   or vice versa
6. step scaling with SF coupling

Combine SF and GF running couplings



1. physical scale $L_{\rm max}$ from LV (CLS) runs
   [LW action, $m_\pi \neq 0$, $m_{\rm K} \neq 0$]

   $$\bar{g}^2_{\rm GF}(L_{\rm max}, c, \varphi)$$

   (maybe massive scheme?)
2. step scaling $L_{\rm max} \to L_{\rm max}/2 \to \dots$
   (change to massless scheme?)

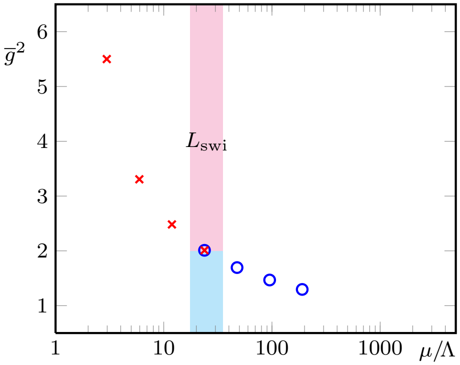
   $$\bar{g}^2_{\rm GF}(L_{\rm max}/4, c, 0)$$

3. step scaling $L_{\rm max}/4 \to L_{\rm max}/8 \to \dots$
4. precision alert at some scale $L_{\rm switch}$
5. match $\bar{g}^2_{\rm SF}$ & $\bar{g}^2_{\rm GF}$ NP'ly, fix $u = \bar{g}^2_{\rm GF}(L_{\rm swi})$

   $$\bar{g}^2_{\rm SF}(L_{\rm swi}) = \lim_{a/L\to 0}\Psi(u, a/L)$$

   or vice versa
6. step scaling with SF coupling

Combine SF and GF running couplings



1. physical scale $L_{\rm max}$ from LV (CLS) runs
   [LW action, $m_\pi \neq 0$, $m_{\rm K} \neq 0$]

   $$\bar{g}^2_{\rm GF}(L_{\rm max}, c, \varphi)$$

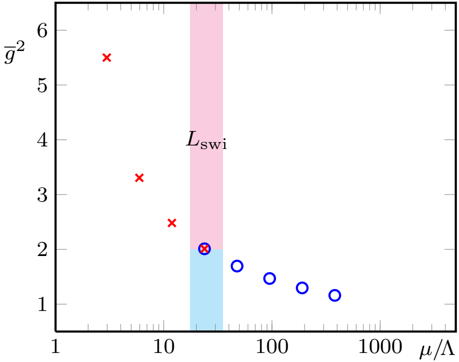
   (maybe massive scheme?)
2. step scaling $L_{\rm max} \to L_{\rm max}/2 \to \dots$
   (change to massless scheme?)

   $$\bar{g}^2_{\rm GF}(L_{\rm max}/4, c, 0)$$

3. step scaling $L_{\rm max}/4 \to L_{\rm max}/8 \to \dots$
4. precision alert at some scale $L_{\rm switch}$
5. match $\bar{g}^2_{\rm SF}$ & $\bar{g}^2_{\rm GF}$ NP'ly, fix $u = \bar{g}^2_{\rm GF}(L_{\rm swi})$

   $$\bar{g}^2_{\rm SF}(L_{\rm swi}) = \lim_{a/L\to 0} \Psi(u, a/L)$$

   or vice versa
6. step scaling with SF coupling



1. physical scale $L_{\rm max}$ from LV (CLS) runs
   [LW action, $m_\pi \neq 0$, $m_{\rm K} \neq 0$]

   $$\bar{g}^2_{\rm GF}(L_{\rm max}, c, \varphi)$$

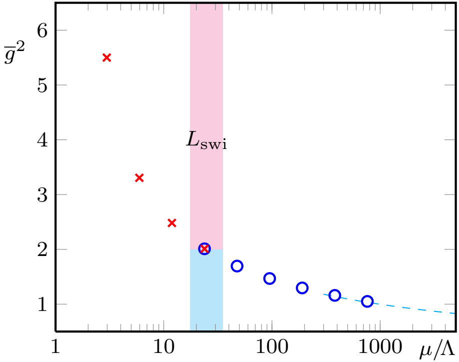
   (maybe massive scheme?)
2. step scaling $L_{\rm max} \to L_{\rm max}/2 \to \ldots$
   (change to massless scheme?)

   $$\bar{g}^2_{\rm GF}(L_{\rm max}/4, c, 0)$$
3. step scaling $L_{\rm max}/4 \to L_{\rm max}/8 \to \ldots$
4. precision alert at some scale $L_{\rm switch}$
5. match $\bar{g}^2_{\rm SF}$ & $\bar{g}^2_{\rm GF}$ NP'ly, fix $u = \bar{g}^2_{\rm GF}(L_{\rm swi})$

   $$\bar{g}^2_{\rm SF}(L_{\rm swi}) = \lim_{a/L \to 0} \Psi(u, a/L)$$

   or vice versa
6. step scaling with SF coupling

Combine SF and GF running couplings



6+7 finished soon

1. physical scale $L_{\rm max}$ from LV (CLS) runs
   [LW action, $m_\pi \neq 0$, $m_{\rm K} \neq 0$]
   $$\bar{g}^2_{\rm GF}(L_{\rm max}, c, \varphi)$$
   (maybe massive scheme?)
2. step scaling $L_{\rm max} \to L_{\rm max}/2 \to \ldots$
   (change to massless scheme?)
   $$\bar{g}^2_{\rm GF}(L_{\rm max}/4, c, 0)$$
3. step scaling $L_{\rm max}/4 \to L_{\rm max}/8 \to \ldots$
4. precision alert at some scale $L_{\rm switch}$
5. match $\bar{g}^2_{\rm SF}$ & $\bar{g}^2_{\rm GF}$ NP'ly, fix $u = \bar{g}^2_{\rm GF}(L_{\rm swi})$
   $$\bar{g}^2_{\rm SF}(L_{\rm swi}) = \lim_{a/L\to 0}\Psi(u, a/L)$$
   or vice versa
6. step scaling with SF coupling
7. connect $\bar{g}^2_{\rm SF}$ to PT at high energies

PRELIMINARY RESULTS

Setting up a line of constant physics

for $N_\mathrm{f} = 3$ degenerate flavours of massless quarks

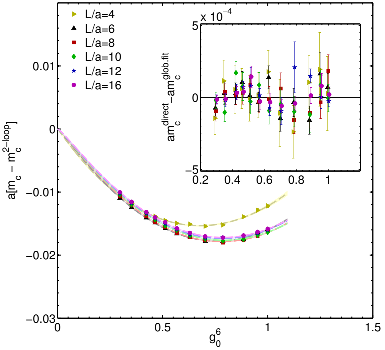
Choose input parameters $(T/L, L/a, \beta, \{\kappa_i\})$ + boundary conditions of the fields:

- $L/a \in \{4, 6, 8, 10, 12\},\ T/L = 1,\ \beta = 6/g_0^2$ + non-vanishing SF b.c.
- $\kappa_{1,2,3} = \kappa \;\Leftrightarrow$ bare current quark mass am

Choose input parameters $(T/L, L/a, \beta, \{\kappa_i\})$ + boundary conditions of the fields:

- $L/a \in \{4, 6, 8, 10, 12\},\ T/L = 1,\ \beta = 6/g_0^2$ + non-vanishing SF b.c.
- $\kappa_{1,2,3} = \kappa \ \Leftrightarrow$ bare current quark mass am

tune to vanishing mass $Lm = 0$ $\Leftrightarrow$ 'critical' quark mass $m_0 = m_c$



60 simulations $\forall L/a$ $(6 \leq \beta \leq 9,\ |Lm| < 0.005)$

$\Downarrow$

$\rightsquigarrow$ smooth function $\forall\, g_0^2 \leq 1$

$$am_c(g_0^2, a/L) = [am_c(g_0^2, a/L)]_{\text{2-lp}} + k_1(a/L)\cdot g_0^6 + k_2(a/L)\cdot g_0^8 + k_3(a/L)\cdot g_0^{10}$$

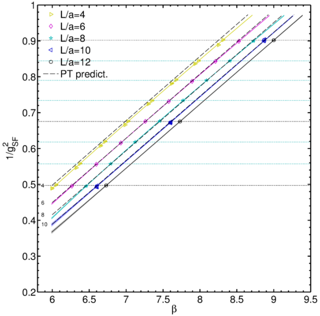
$\Downarrow$

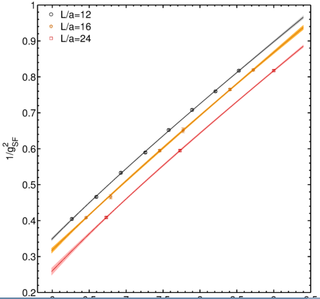
$|Lm_c| < 0.001$

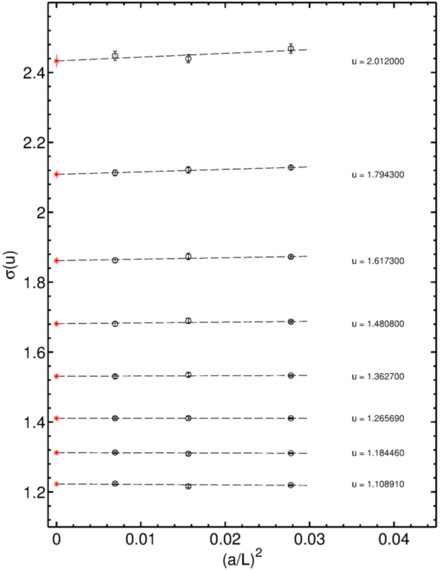
additional uncertainty from LCP tuning *negligible*

Step-scaling function for $\bar{g}^2_{\mathrm{SF}}$

preliminary







Summary

increase knowledge of $\alpha_s(\mu) \leftrightarrow \Lambda$

- connect hadronic low-energy and high-energy regime
  $\rightsquigarrow$ tests of QCD on scale separations of order $\gtrsim 10^3$
- systematic uncertainties well under control using
  lattice simulations + finite-size scaling + physical running couplings
- PT invoked at very high energies only $\gtrsim 100\,\mathrm{GeV}$
- best to combine running of $\bar{g}^2_{\mathrm{SF}}$ and $\bar{g}^2_{\mathrm{GF}}$ non-perturbatively

Gradient flow

new tool to probe non-perturbative aspects of strongly coupled gauge theories

- solid theoretical understanding
- GF coupling advantageous to reach even lower scales
- wide range of applications
  - simplified renormalization
  - fermion flow [Lüscher:'13]: no operator mixing on the lattice
  - conformal field theories, walking technicolor, . . .

we are in a good position to achieve $\Delta\Lambda/\Lambda \sim 2\%$

THANK YOU FOR YOUR ATTENTION!

And many thanks to my collaborators:

M. Dalla Brida
T. Korzec
A. Ramos
S. Sint
R. Sommer
U. Wolff