Remarks on the sign problem of worldline fermions

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by Ulli Wolff

Humboldt Universität, Berlin

Plan and overview

- warning: dissection of the problem, no solution and not new (2009)
- simple example:
  free Majorana-Wilson lattice fermions in Euclidean $D = 2, 3$
- worldline or all-order hopping parameter graph representation
- no sign problem in $D = 2$ ($\Longrightarrow$ Thirring, Gross Neveu model)
- severe sign problem in $D = 3$
- some geometric understanding of oscillating phase
- suitable testbed to solve such sign problems?

Majorana-Wilson in $D = 2, 3$

Dirac: $\psi = (\psi_1, \psi_2)^\top$, $\overline{\psi} = (\overline{\psi}_1, \overline{\psi}_2)$ independent Grassmann

$$\psi = \frac{1}{\sqrt{2}}(\xi + i\eta), \quad \overline{\psi} = \frac{1}{\sqrt{2}}(\xi^\top - i\eta^\top)\mathcal{C}, \quad \xi = (\xi_1, \xi_2)^\top, \quad \eta = (\eta_1, \eta_2)^\top$$

charge conjugation matrix $\mathcal{C}$:

$$\mathcal{C}\gamma_\mu\mathcal{C}^{-1} = -\gamma_\mu^\top, \quad \mathcal{C}^\top = -\mathcal{C}$$

decompose action

$$\overline{\psi}\left(\gamma_\mu\tilde{\partial}_\mu + m - \tfrac{a}{2}\partial_\mu\partial_\mu^*\right)\psi = \tfrac{1}{2}\xi^\top\mathcal{C}\left(\gamma_\mu\tilde{\partial}_\mu + m - \tfrac{a}{2}\partial_\mu\partial_\mu^*\right)\xi + (\xi \to \eta)$$

$\Longrightarrow$ simplest unit is ξ only, shorthand $\bar{\xi} := \xi^\top\mathcal{C}$ (not independent)

- 2 Grassmann components per site

$$S = a^D \frac{1}{2} \sum_x \bar{\xi}\left(\gamma_\mu\tilde{\partial}_\mu + m - \tfrac{a}{2}\partial_\mu\partial_\mu^*\right)\xi$$

- [$m \to m + \sigma(x)$ $\to$ integrate $\to$ interaction, e.g. $(\bar{\xi}\xi)^2$, add flavor]

Loop representation

$$Z = \int D\xi \, \mathrm{e}^{-S} = \int D\xi \left[\prod_z \mathrm{e}^{-\frac{1}{2}(D+m)\bar{\xi}\xi} \right] \prod_{\langle x\, y \rangle} \mathrm{e}^{\bar{\xi}\,(x) P(y-x) \xi(y)}$$

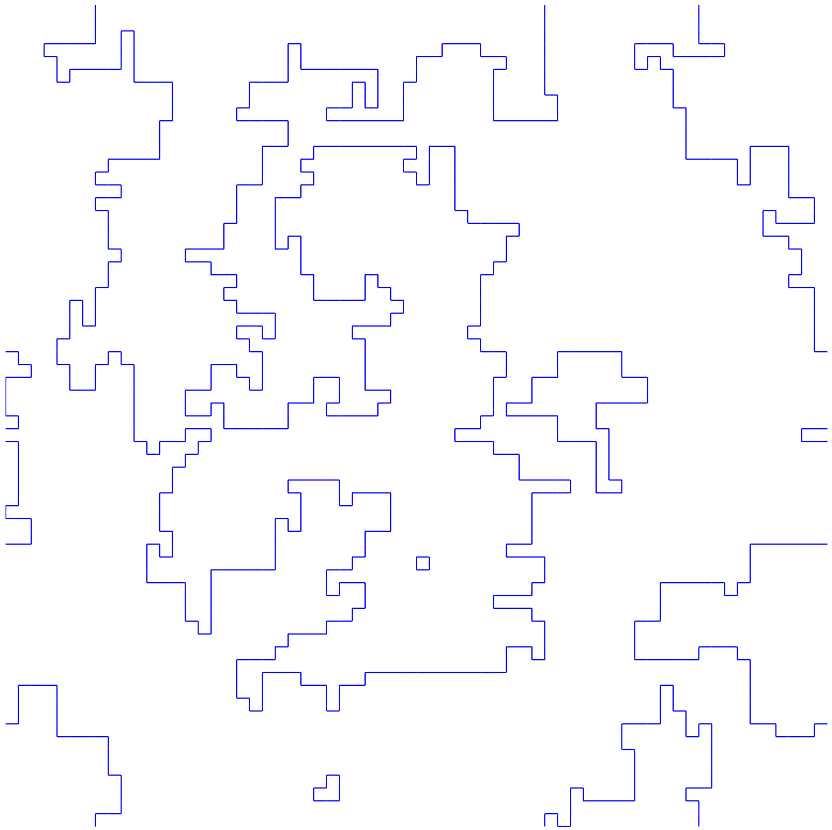
with $a = 1$, and projectors (for any direction $n_\mu n_\mu = 1$)

$$P(n) = \frac{1}{2}(1 - n_\mu \gamma_\mu)$$

$$\mathrm{e}^{\bar{\xi}\,(x) P(y-x) \xi(y)} = 1 + \bar{\xi}\,(x) P(y-x) \xi(y) = \sum_{k_{x\,y}=0,1} [\bar{\xi}\,(x) P(y-x) \xi(y)]^{k_{x\,y}}$$

$\leftarrow$ nilpotent due to Grassmann and 1d projector P

- insert expansion with $k_{x\,y}$=0 or $k_{x\,y}$=1 on all links
- for given values $\{k_{x\,y}\}$ perform $\int D\xi \ldots$
- contribution $\neq 0$ to Z $\Leftrightarrow$ ZERO or TWO $k_{x\,y} = 1$ adjacent at each site
- graphs on the lattice: line on link $\langle x\,y \rangle$ $\leftrightarrow k_{x\,y} = 1$
  $Z = \sum$ gas of closed non-intersecting loops with certain weights



- $\int d^2\xi\, \xi_\alpha \bar{\xi}_\beta = \delta_{\alpha\beta}$ at each site leads to:
- empty sites: factors $D + m$ $\quad(\int \; \ldots .$ saturated by $\mathrm{e}^{-\frac{1}{2}(D+m)\bar{\xi}\xi})$
- factor for each loop: $w(l) = -\,\mathrm{tr}[P(n_1)P(n_2) \times \cdots \times P(n_{|l|})]$
- (unoriented) loop given by sequence of unit 'steps' n_i ($\Rightarrow \sum_i n_i = 0$)
- no other signs [move around binlinears in monomials only]

The loop factor

see also Nucu Stamatecu, PRD25(1982)1130

$$w(l) = -\,\mathrm{tr}[P(n_1)P(n_2)\times\cdots\times P(n_{|l|})]$$

$$P(n_1)P(n_2) = R_{12}P(n_2)R_{12}^{-1}P(n_2), \quad R_{12} = \mathrm{e}^{\frac{\pi}{8}[\not{n}_1,\not{n}_2]} \quad \text{rotates}\, n_2 \to n_1$$

$$P(n_2)R_{12}^{-1}P(n_2) = P(n_2)\times\cos\!\left(\frac{1}{2}\sphericalangle(n_1,n_2)\right) \in (1, 2^{-1/2})$$

$$P(n_1)P(n_2)P(n_3) = R_{12}R_{23}P(n_3)\cos\!\left(\frac{1}{2}\sphericalangle(n_1,n_2)\right)\cos\!\left(\frac{1}{2}\sphericalangle(n_2,n_3)\right)$$

$$\cdots\cdots$$

$$w(l) = -\,2^{-\#\text{corners}/2}\,\mathrm{tr}[R_{\text{loop}}P(n_1)], \quad R_{\text{loop}} = R_{12}R_{23}\cdots R_{|l|-1\,|l|}R_{|l|1}$$

about R_{loop}:

- is in the spinor repres. $\Rightarrow \pm R$ is one rotation ($R = -1 \to 2\pi$ rot.)
- rotates n_1 into itself
  - $D = 2$: $R_{\text{loop}} = (-1)^{\#\,\text{of turns of loop}}$
  - $D = 3$: $R_{\text{loop}} =$ rotation around axis n_1 by some angle

$D = 2$

- topologically trivial loop (not winding around torus) has: ONE turn

$$w(l) = +\,2^{-\#\mathrm{corners}/2} = \mathrm{e}^{-\mu\#\mathrm{corners}} \qquad \mu = \ln(2)/2$$

- Fermi minus $\times$ spinor 2π minus cancel loop by loop
- Hamiltonian analog (?): Jordan Wigner trafo stays local

- this loop gas has a continuum limit (free Majorana)

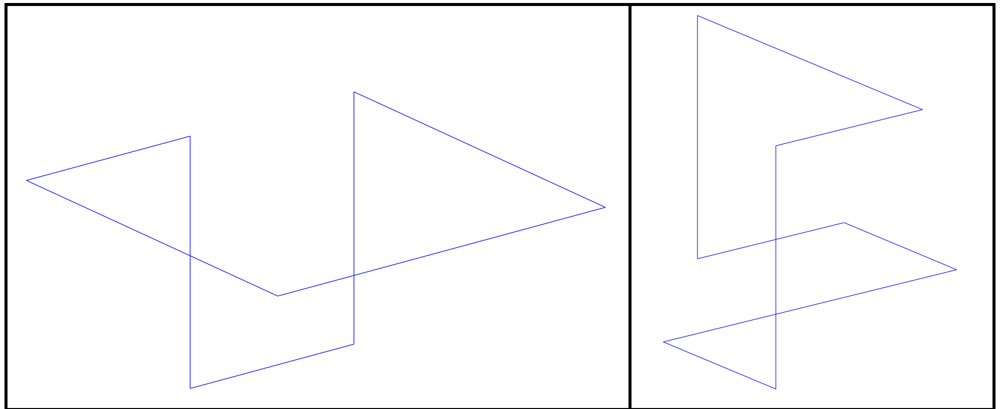
- only topologically trivial loops for $mL \gg 1$
- general case sign $\leftrightarrow$ Z(2) winding number of loops
- with certain boundary conditions (p./a.p.) exact positivity for any mL

- Worm simulation with extra insertion $\xi_\alpha(u)\bar{\xi}_\beta(v) \rightarrow$ pleasant CSD

$D=3$

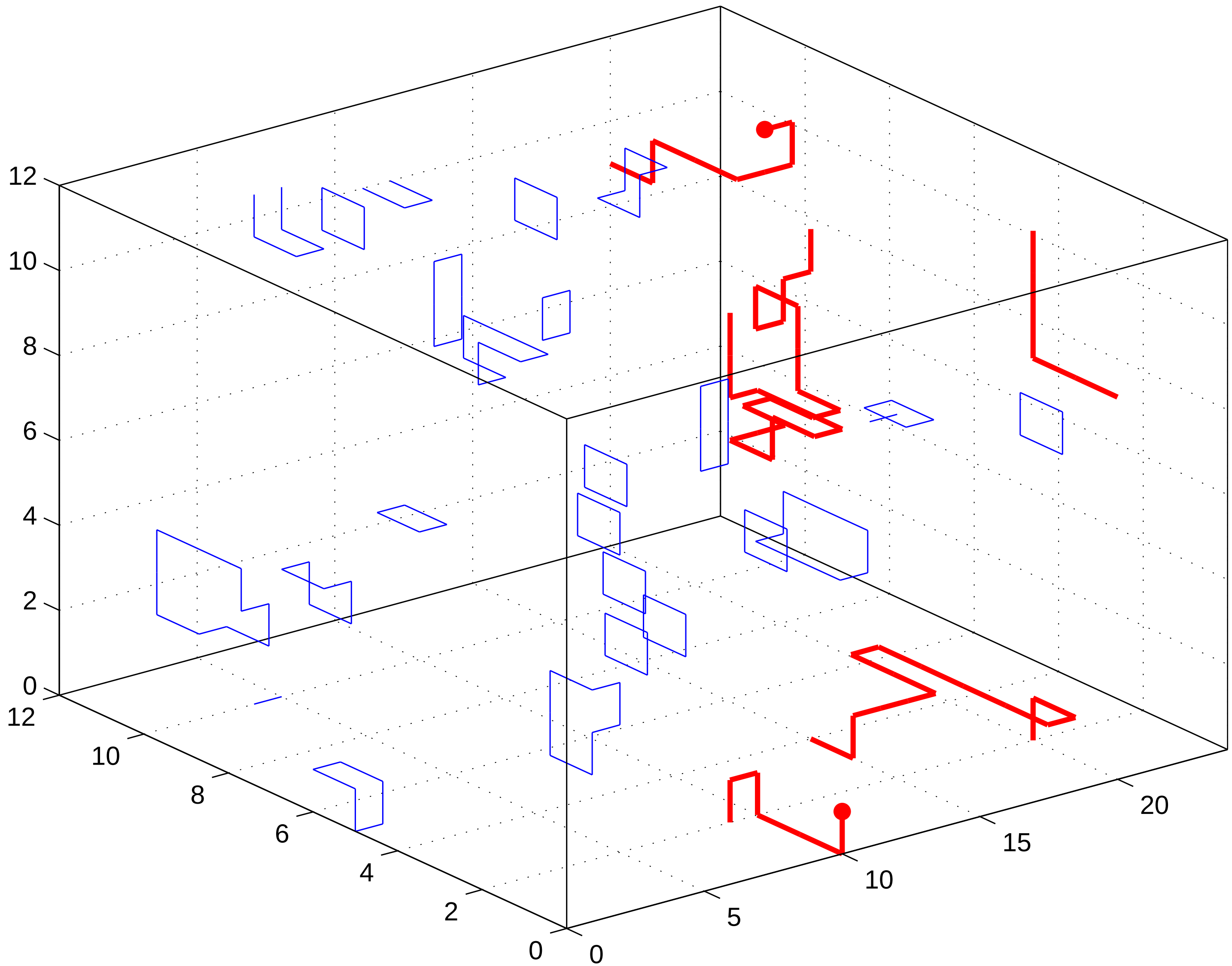
$$R_{\text{loop}} = \mathrm{e}^{i\frac{\alpha}{2}\not{n}_1}, \quad \mathrm{e}^{i\alpha} \in Z(8)$$

- Z(8) spinor covering of cubic planar rotations
- $w(l)$ still positive for planar loops
- but: all phases appear for small nonplanar loops:



$w \propto \exp(i\pi/4)$ | $w \propto \exp(i\pi/2)$

- sign reweighting okay for $am > 1/2$ (correct correlation seen)
- planar (lattice artefact-) dominated
- variance explodes for $am \lessapprox 1/2$
- this loop gas has a (non-positive) continuum limit (free Majorana)



$am = 0.65$ on 24×12^2

Continuum limit of the loop phase

- closed loop in $D=3$: $\vec{\gamma}(t), t \in [0,1]$ periodic
- velocity $n(t) = \dot{\vec{\gamma}}/|\dot{\vec{\gamma}}| \Rightarrow n^2 = 1$, $\vec{n}(t)$ obeys ($\not{n} = n_\mu \gamma_\mu$)

$$\dot{\not{n}} = \dot{\not{n}}\not{n}^2 = \frac{1}{2}[\dot{\not{n}}\not{n}\,, \not{n}]$$

solve for spinor rotation

$$\frac{d}{dt}R(t) = \frac{1}{2}\dot{\not{n}}\not{n}\, R(t), \quad R(0) = 1$$

such that

$$\not{n}(t) = R(t)\not{n}(0)R^{-1}(t)$$

- $\not{n}(1) = \not{n}(0)$

$$\Longrightarrow R(1) = \mathrm{e}^{i\frac{\alpha}{2}\not{n}(0)}, \quad \mathrm{e}^{i\alpha} \in U(1)$$

valid for lattice and continuum: $\mathrm{e}^{i\alpha} \in Z(8), U(1)$ is

- independent of where the loop is cut/started
- independent of the direction of transversal (Majorana)
- i.e. a property of each unoriented loop

Conclusions & tasks

- fermion $\leftrightarrow$ loop gas in a simple case $D = 2$ and $D = 3$

- $D = 2$ no sign problem
- $D = 3$ strong sign problem, simulation only for $\xi_{\mathrm{corr}} < 2$
- look for remedies
  - combine terms analytically (meron algo)
  - add extra variables (cluster methods)
  - ???
- testbed!

- extend: Majorana + $Z(2)$ gauge fields, combine with surface formulation/simulation of (Abelian so-far) gauge theory
- extend: Dirac + $U(1)$ gauge field, Schwinger