Studying the gradient flow coupling in the SF

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IN COLLABORATION WITH ALBERTO RAMOS

BASED ON [ARXIV:1301.4388]

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Motivation

Determine the energy scale of asymptotically free theories: $\Lambda-$parameter

Basic ingredients:

1. Non-perturbative definition of the running coupling — lattice as regulator
2. Safely bridge the gap between non-perturbative and perturbative energy scales via step-scaling method — Schrödinger functional
3. High-precision computation of the renormalized coupling in the continuum limit — Wilson flow

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Basic obstacles:

- CONTINUUM EXTRAPOLATION
  cutoff effects $\longleftarrow$ lattice action; scaling behaviour; ...
- CONNECTION TO PT AT HIGH ENERGIES
  knowledge of PT coeff.s depends on chosen scheme
- COMPUTATIONAL EFFORT
  scaling of algorithm (HMC, ...) & autocorrelations; tuning $\longleftarrow L, a, m_{\mathrm{q}}, \ldots$, run parameters

The gradient flow

(or Wilson flow) [Lüscher:'09-'13]

flow $B_\mu(x,t)$ of $SU(N)$ gauge fields driven by $(t \geq 0)$: $x = (x_0, \mathbf{x})$

$$\frac{\mathrm{d}B_\mu(x,t)}{\mathrm{d}t} = D_\nu G_{\nu\mu}(x,t)\,, \qquad \left(\propto -\frac{\delta S_{\mathrm{YM}}[B]}{\delta B_\mu}\right)$$

$$B_\mu(x,t)|_{t=0} = A_\mu(x)\,, \qquad \text{: initial condition}$$

$$G_{\mu\nu}(x,t) = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]\,, \qquad D_\nu = \partial_\nu + [B_\nu, \star]$$

Properties:

- continuous smoothing of gauge fields with flow time t
- UV finite for $t > 0$, proven to all orders in PT for YM theory [Lüscher,Weisz:'11]
  $\Rightarrow \exists$ continuum limit for purely gluonic observables since B_μ are renormalized fields
- excellent numerical precision
- ...

- *future applications*, M.Lüscher (plenary talk; tomorrow)
- A.Ramos, *The gradient flow in a twisted box* (Parallels 6E)

The energy density

as most exploited quantity so far

[Lüscher:'10]

for $SU(N)$, $D = 4$, $V = \infty$

$$\langle E(t)\rangle \equiv \frac{1}{4}\langle G_{\mu\nu}(t) G_{\mu\nu}(t)\rangle = \frac{3(N^2-1)}{2(8\pi t)^2} \times \bar{g}^2_{\mathrm{MS}}(\mu)\Big\{1 + c_1 \bar{g}^2_{\mathrm{MS}} + \mathrm{O}\big(\bar{g}^4_{\mathrm{MS}}\big)\Big\}$$

and $\mu = 1/\sqrt{8t}$

- $\sqrt{8t}$ is effective smoothing radius of Wilson flow
- automatic renormalized at $t > 0$

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- provides non-perturbative definition of a 'gradient flow' coupling in finite-volume $L^3 \times T$

$$t^2\langle E(t)\rangle = \mathcal{N}\cdot\bar{g}^2(\mu)\ , \qquad \mu^{-1} = \sqrt{8t} = cL\ , \qquad \forall N, N_{\mathrm{f}}$$

  - $\mathcal{N}$ normalization factor such that $\bar{g}^2 = g_0^2 + \mathrm{O}\left(g_0^4\right)$
  - $c = \sqrt{8t}/L$ effective smoothing range of Wilson flow
  - in finite volume boundary conditions become important

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1^{st} computation by [FodorEtAl:'12] using periodic boundary conditions

$\Rightarrow$ PT in powers of g_0, NOT g_0^2: no universal 2-loop coeff. in $\beta-$function

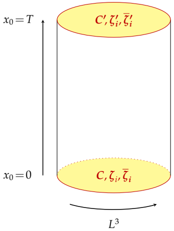
absent by using Schrödinger functional boundary conditions

- Euclidean partition function

$$\mathcal{Z} \equiv \int_{T\times L^3} \mathcal{D}[U,\overline{\psi},\psi]\,\mathrm{e}^{-S[U,\overline{\psi},\psi]} = \langle 0|\,\mathrm{e}^{-TH}P|0\rangle$$

  with *periodic BC in L^3*
  and *Dirichlet BC in T* (breaking translational inv. in time)

- renormalization scale $\mu \propto L^{-1}$ (for step-scaling)
- ...



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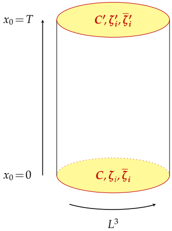

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for spatially Fourier transformed flow field the boundary conditions read

$$\forall \mathbf{p}: \qquad \tilde{B}_k(\mathbf{p},x_0,t)\big|_{x_0=0,T} = 0\,,$$

while behaviour of time component, $\tilde{B}_0$, emerges through gauge fixing,

$$\mathbf{p}\neq 0: \qquad \partial_0\tilde{B}_0(\mathbf{p},x_0,t)\big|_{x_0=0,T} = 0\,,$$

$$\mathbf{p}= 0: \qquad \tilde{B}_0(\mathbf{0},x_0,t)\big|_{x_0=0} = 0\,, \qquad \partial_0\tilde{B}_0(\mathbf{0},x_0,t)\big|_{x_0=T} = 0\,.$$

Computing the normalization factor $\mathcal{N}$

The energy density in perturbation theory

rescaling & expanding the gauge fields at $t > 0$

$$B_\mu = \sum_{n=1}^{\infty} B_{\mu,n} g_0^n \ , \qquad \langle E(t,x_0)\rangle = \sum_{n=0}^{\infty} \mathcal{E}_n(t,x_0) \ , \ \text{with} \quad \mathcal{E}_n = \mathrm{O}\big(g_0^{2+n}\big)$$

LO: $$\mathcal{E}_0(t,x_0) = \frac{g_0^2}{2}\langle \partial_\mu B^a_{\nu,1}\partial_\mu B^a_{\nu,1} - \partial_\mu B^a_{\nu,1}\partial_\nu B^a_{\mu,1}\rangle$$

expanding flow equation $\longrightarrow$ tower of equations

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LO: Flow equation = Heat equation $\quad \dot{B}_{\mu,1} = \partial_\nu \partial_\nu B_{\mu,1}$ + gauge terms

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{B}_{\mu,1}(\mathbf{p},x_0,t) = (-\mathbf{p}^2 + \partial_0^2)\tilde{B}_{\mu,1}(\mathbf{p},x_0,t) \ , \qquad \tilde{B}_{\mu,1}(\mathbf{p},x_0,t)\big|_{t=0} = \tilde{A}_\mu(\mathbf{p},x_0)$$

- heat kernels need to comply with boundary conditions $\rightarrow K^D, K^N, K^{ND}, \ldots$

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$\downarrow$ LO

$$\langle E(t)\rangle = \mathcal{E}_0(t) = \partial\partial \int [\text{heat kernel stuff}]\langle AA\rangle \qquad \Rightarrow \mathcal{N}(c, T/L, x_0/T)$$

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computation of lattice norm along the same lines: $\qquad \Rightarrow \widehat{\mathcal{N}}(c, T/L, x_0/T, a/L)$

Non-perturbative tests

Line of Constant Physics (LCP)

defined by traditional SF coupling ($N_\mathrm{f} = 2,\, L_1 \sim 0.4\,\mathrm{fm}$)

$$\bar{g}^2_{\mathrm{SF}}(L_1) \equiv u = 4.484 \qquad \text{and} \qquad m(L_1) = 0\;,$$

for lattice sizes $L/a = 6, 8, 10, 12, 16$ with $T = L$

Our observable, defined at $x_0/T = 1/2$, $T/L = 1$, thus reads

$$\Omega(u; c, a/L) = \left[\hat{\mathcal{N}}^{-1}(c, T/L, x_0/T, a/L) \cdot t^2 \langle E(t, T/2)\rangle\right]^{\mathrm{LCP}}_{t=c^2L^2/8}$$

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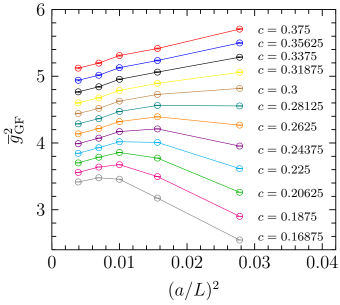
data shown:

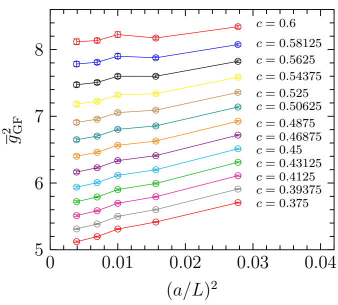
$c \geq c_{\min}$

$c_{\min} = \max(\{a/L\})$,

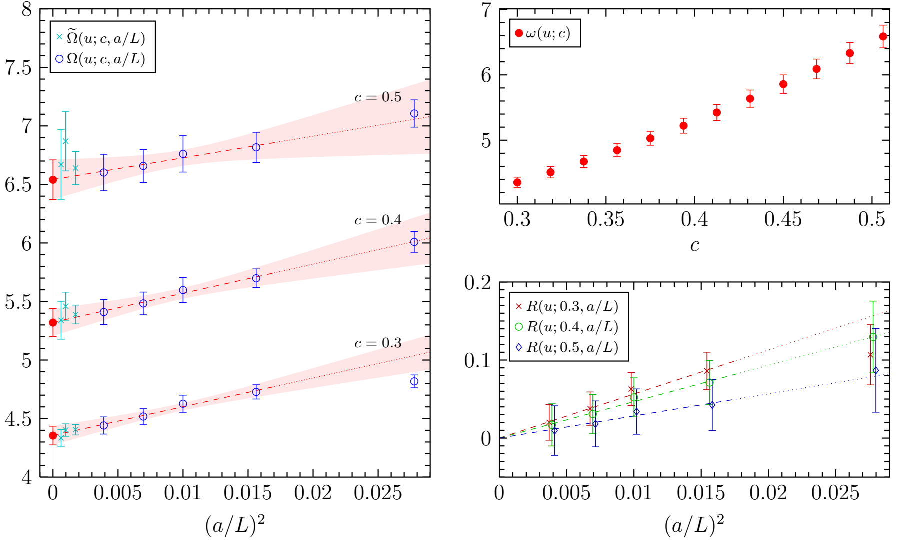
$\forall L/a$

error from $\overline{g}^2_{\mathrm{SF}}$ not propagated here





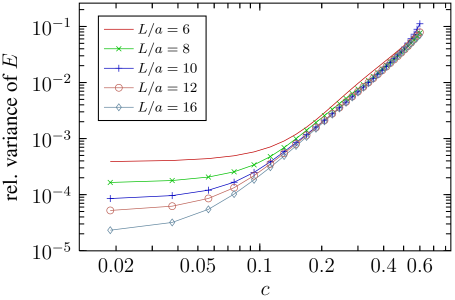
Taking the continuum limit (c fixed) after including uncertainty from $\overline{g}^2_{\mathrm{SF}}$



Relative variance of observable $\mathcal{O}$

$$\mathcal{V}_{\mathcal{O}} = \frac{\langle \mathcal{O}^2 \rangle - \langle \mathcal{O} \rangle^2}{\langle \mathcal{O} \rangle^2}$$

$\mathcal{V}_{\bar{g}^2_{\mathrm{GF}}} = \mathcal{V}_E$:



The SF gradient flow coupling

Cost figure, Variance

[arXiv:1301.4388]

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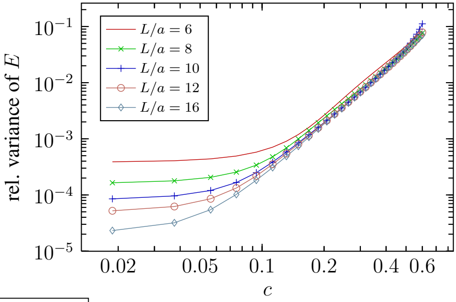
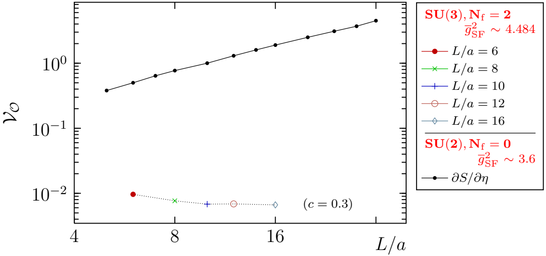


comparison $\mathcal{V}_E$ vs. $\mathcal{V}_{\partial S/\partial\eta}$



quenched $SU(2)$ study [deDiviitisEtAl:'95]:

$\mathcal{V}_{\partial S/\partial\eta}$ of traditional SF coupling
diverges

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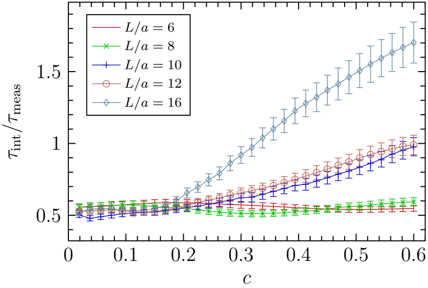
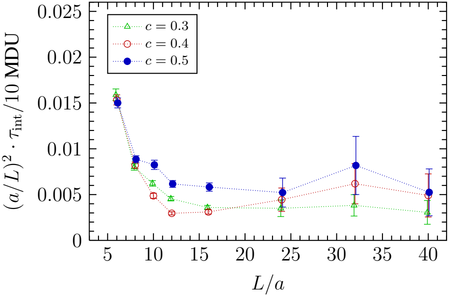
Cost figure, Autocorrelations

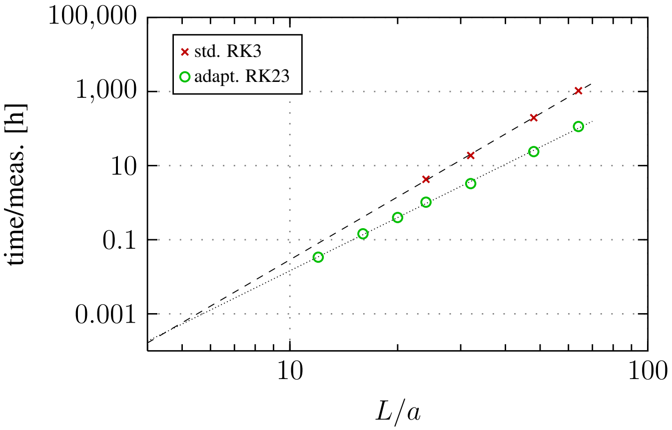
[arXiv:1301.4388]





$$\tau_{\rm int} \propto \frac{1}{a^2}$$

gain by using adaptive step-size integrator:



Summary and outlook

- combination Lattice + SF + Wilson flow $\longrightarrow$

  new NP definition of a running coupling in asymptotically free theories

  $\rightarrow$ potentially smaller error on $\Lambda_{\overline{\mathrm{MS}}}$

- dependence on sea quark mass seems to be smaller than for $\bar{g}^2_{\mathrm{SF}}$ $\longrightarrow$

  tuning to vanishing quark mass get's easier ?

- improved signal-to-noise ratio $\longrightarrow$ more useful in beyond the SM theories ?

- full setup should be chosen carefully: many things to investigate

  - lattice action $\leftarrow c_{\mathrm{sw}}, c_{\mathrm{t}}, \widetilde{c}_{\mathrm{t}}, \ldots$ / accuracy of conversion functions, ...
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my sincere appologies for not talking about our

Preparations for $N_{\rm f} = 3$ simulations:

- investigating $c_{\rm t}$ & $\widetilde{c}_{\rm t}$ dependence of $\bar{g}^2_{\rm GF}$
- step-scaling function in $N_{\rm f} = 0$ QCD at $\bar{g}^2_{\rm GF} \sim 1.0, 2.9$ for $L/a = 8, 10, 12, 16, 20, 24, 32$
  using both Wilson plaquette and tree-level improved Lüscher-Weisz gauge action

Thanks go to: A.Ramos, R.Sommer, U.Wolff, M.Lüscher and many colleagues @DESY,HU
...and you for listening