Taking worm algorithms from spin models to Abelian lattice gauge theory

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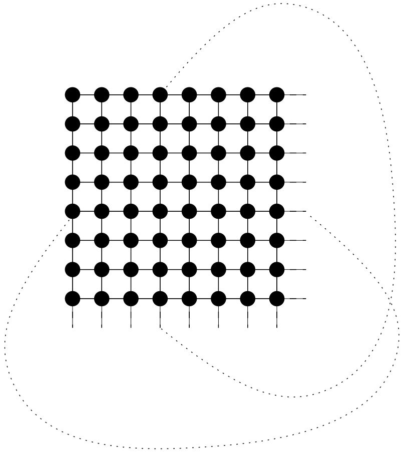
Subjects touched upon:

- Ising model = prototype model for
  - almost all concepts in statistical physics, e.g. phase transitions, universality,...
  - simplest lattice quantum field theory (imaginary time), scalar particles, spontaneous symmetry breaking, "Higgs"
- Monte Carlo simulation, standard and worm
  - in principle exact up to statistical errors, alternatives:
  - exact solution in $D \leqslant 2$ only (Onsager)
  - systematic weak/strong coupling (= low/high temperature) expansion series (truncated!)
- Ising spin model $\rightarrow$ Ising lattice gauge theory (gauge theory$\leftrightarrow$particle physics, standard model)

Ising model, our notation



- hypercubic torus, D dimensions
- sites x, directions μ
- spin configurations $s \equiv \{s(x) = \pm 1\}$

Euclidean action/Hamiltonian:

$$-S(s) = \beta \sum_{x\mu} s(x)s(x+\hat{\mu})$$

partition function:

$$Z_0 = \sum_s \mathrm{e}^{-S(s)} \quad \rightarrow \quad Z_2(u,v) = \sum_s s(u)s(v)\mathrm{e}^{-S(s)}$$

fundamental correlation:

$$\langle s(u)s(v) \rangle = \frac{Z_2(u,v)}{Z_0} = G(u-v)$$

Monte Carlo method

$$\langle \mathcal{O} \rangle = \sum_s \underbrace{P(s)}_{e^{-S(s)}/Z_0} \mathcal{O}(s) \qquad [\text{example: } \mathcal{O}(s) = s(u)s(v)]$$

- draw $s^{(1)}, s^{(2)}, \ldots, s^{(N)}$ each with probability $P(s)$
- estimate $\langle \mathcal{O} \rangle \simeq \frac{1}{N}\sum_{i=1}^{N} \mathcal{O}(s^{(i)})$
- error $\propto 1/\sqrt{N}$

problem:

- no practicable method to independently draw with $P(s)$ for large lattices at $\beta \approx \beta_{\text{critical}}$
- generate $s^{(i)} \to s^{(i+1)}$ in a Markov chain, disadvantage:
- autocorrelations, error has large (diverging) prefactor (CSD)

in addition:

$\langle s(u)s(v) \rangle \propto \exp[-|u-v|/\xi], \quad \langle [s(u)s(v)]^2 \rangle = 1,$ bad signal/noise

All order strong coupling reformulation

- $Z_0, Z_2\,(, Z_4\ldots)$ have expansions in β
- convergent for all β in a finite volume
- this includes $\beta \approx \beta_c,\, \xi \gg 1$
- but: contributions $\sim \beta^{\text{volume}}$ will be important!
- [normal (truncated) s.c.: $V \to \infty$ term by term in Z_2/Z_0]

expand for each link independently:

$$\mathrm{e}^{\beta\sigma(x)\sigma(x+\hat{\mu})} = \sum_{k=0}^{\infty} \frac{\beta^k}{k!}\,\sigma(x)^k \sigma(x+\hat{\mu})^k \quad [\text{below: } k \to k(x,\mu) \equiv k_\mu(x)]$$

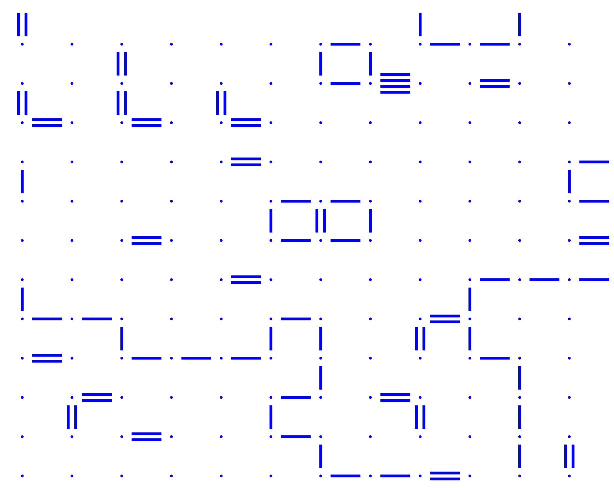
alternative form: $\mathrm{e}^{\beta\sigma(x)\sigma(x+\hat{\mu})} = \cosh\beta \sum_{k=0,1} (\tanh\beta)^k\, \sigma(x)^k \sigma(x+\hat{\mu})^k$

$\Longrightarrow$ use expansion on each link + sum over original spins
[trivial: even power of each $s(x)$ required $\to$ constraints on $\{k(x,\mu)\}$

$$Z_0 = \sum_{g \in \mathcal{G}_0} \beta^{\sum_{x,\mu} k(x,\mu)} W[k]$$

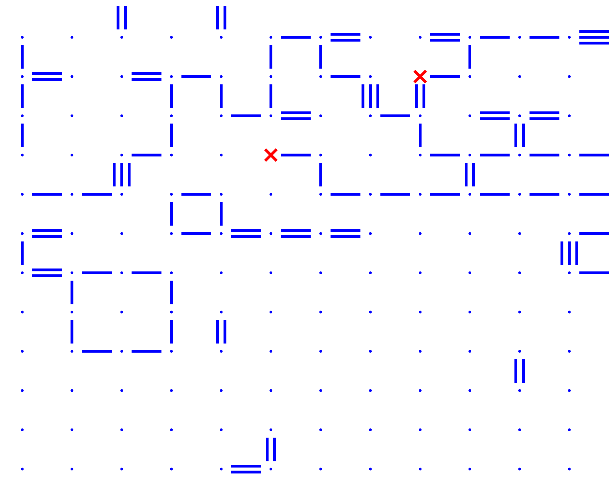
- graphs g with $k(x,\mu) = 0, \ldots, \infty$
- divergence: $\partial^*_\mu k_\mu(x) = 0 \pmod 2$
- $W[k] = \prod_{x,\mu} \frac{1}{k(x,\mu)!}$

$\Rightarrow \beta \langle \sigma\sigma \rangle_{n.n.} = \langle k(x,\mu) \rangle_{g \in \mathcal{G}_0} = \mathrm{O}(1)$



$$Z_2(u,v) = \sum_{g \in \mathcal{G}_{2|u,v}} \beta^{\sum_{x,\mu} k(x,\mu)} W[k]$$

- $\partial^*_\mu k_\mu(x) = \delta_{x,u} + \delta_{x,v} \pmod 2$
- 'defects/sources' at u and v
- $\mathcal{G}_{2|u,u} = \mathcal{G}_0$



The break-through of Prokof'ev and Svistunov

- Z_0 has been simulated as $\sum_{g \in \mathcal{G}_0} \ldots$ in ancient history [Berg & Förster, 1981]
  - $k(x,\mu) \to k(x,\mu) \pm 1$ on small (plaquette) loops
  - additional steps
  - not efficient, critical slowing down

P&S: enlarge the ensemble

$$\mathcal{Z} = \sum_{u,v} Z_2(u,v) = \sum_{g \in \mathcal{G}_2} \beta^{\sum_{x\mu} k(x,\mu)} W[k] \qquad \mathcal{G}_2 = \cup_{u,v} \mathcal{G}_{2|_{u,v}}$$

- PS 'worm' algorithm works on $\mathcal{G}_2$:
  - $k(u,\mu) \to k(u,\mu) \pm 1$ combined with $u \to u + \hat{\mu}$
    [or $k(u-\hat{\mu},\mu) \to k(u-\hat{\mu},\mu) \pm 1$ $u \to u - \hat{\mu}$]
  - a defect moves, constraint preserved
  - (practically) no critical slowing down

- more efficient moves $\mathcal{G}_0 \ni g \to g' \in \mathcal{G}_0$ by cutting through $\mathcal{G}_2$
- the intermediate configurations are extremely useful:

$$G(x) = \langle \sigma(x)\sigma(0) \rangle = \frac{\langle \delta_{x,u-v} \rangle_g}{\langle \delta_{u,v} \rangle_g}, \qquad \langle \delta_{u,v} \rangle_g = \chi^{-1}, \quad \langle . \rangle_g \equiv \langle . \rangle_{g \in \mathcal{G}_2}$$

- all-x 2-point function = histogram $u - v$ of sampled graphs

A very simple generalization $[\rho > 0,\ \rho(0) = 1]$:

$$\mathcal{Z} = \sum_{u,v} Z_2(u,v)\rho^{-1}(u-v) = \sum_{g \in \mathcal{G}_2} \beta^{\sum_l k(x,\mu)} W[k]\rho^{-1}(u-v)$$

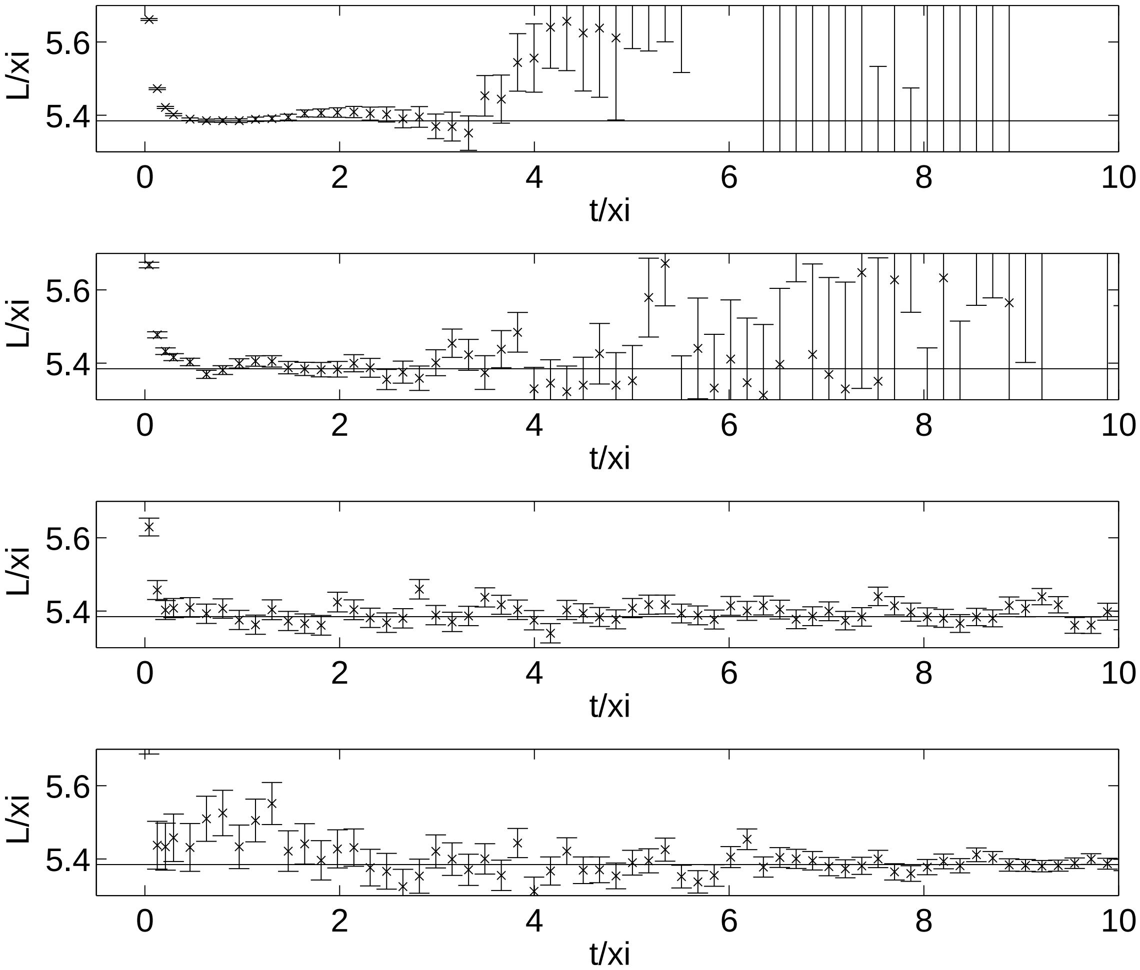
$$G(x) = \langle \sigma(x)\sigma(0) \rangle = \frac{\langle \delta_{x,u-v} \rangle_g}{\langle \delta_{u,v} \rangle_g} \times \rho(x)$$

- use a guess $\rho(x) \approx \langle \sigma(x)\sigma(0) \rangle$
- then $\langle \delta_{x,u-v} \rangle_g$: guess $\rightarrow$exact answer
- $\langle \delta_{x,u-v} \rangle_g \approx$ const $\Rightarrow$all bins $u-v$ get$\approx$same statistics $\Rightarrow$ signal/noise x-independent!

$\xi_{\mathrm{eff}} =$ log-deriv.
$t =$ separation

Ising model on
$L^2 = 64^2$
$\beta = 0.42$
$\xi = 11.88...$
(exact)

[details:
time-slices,
$\exp \to \cosh$]



spin MC

'worm':
$\rho \equiv 1$

$\rho \sim \mathrm{e}^{-t/\xi}$

$\rho \sim \mathrm{e}^{-2.1t/\xi}$

Triviality of φ^4

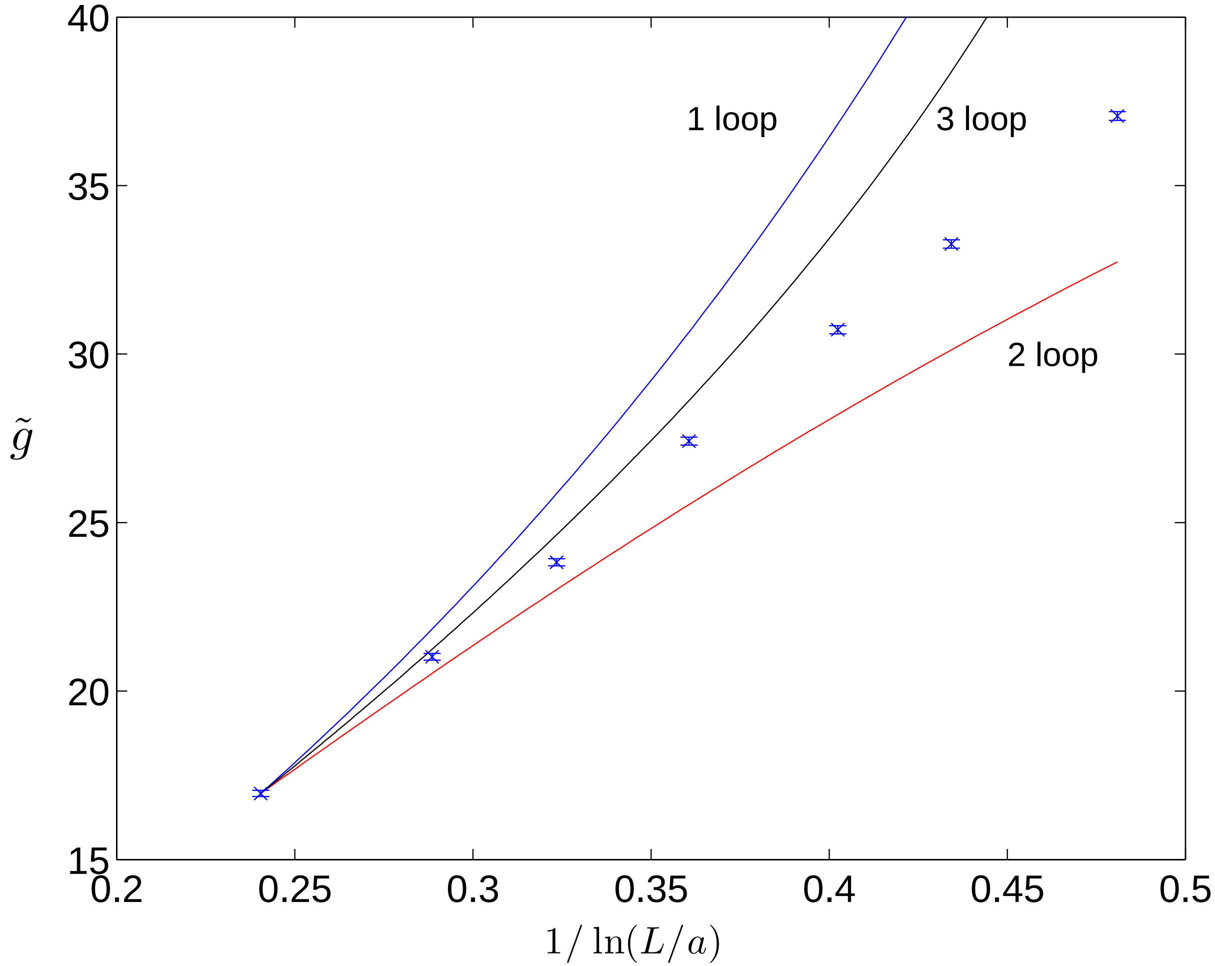
A 'QFT central limit theorem'....

- Aizenman's rigorous proofs (bounds) for $D > 4$ use
  - our $g \in \mathcal{G}_2$ representation for Ising
  - plus: replica and percolation ideas
- Translate into MC estimators for any D (incl. $D = 4$)
- Result

$$g_R = -\frac{\chi_4}{\chi^2}(m_R)^D = 2z^D \langle \mathcal{X} \rangle_{(g,g') \in \mathcal{G}_2 \times \mathcal{G}_2} \quad \mathcal{X} \in \{0, 1\}, z = m_R L$$

($R \leftrightarrow$ renormalized, $\mathcal{X} = 1 \leftrightarrow 4$ defects connected in a bond percolation cluster defined by $k + k' > 0$

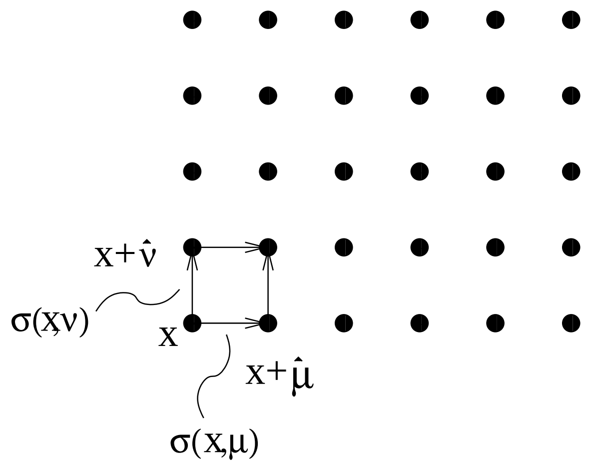
- no numerical cancellation for connected χ_4
- Lebowitz inequality manifest



$$D = 4,\ z = m_R L = 4,\ L/a = 64, \ldots, 8$$

Wegner/Wilson lattice gauge theory

- lattice as before
- Z(2) spin field $s(x) = \pm 1 \rightarrow$ Z(2)-link field $\sigma(x,\mu) \equiv \sigma_\mu(x) = \pm 1$
- gradient coupling $\frac{1}{2}(\partial_\mu s)^2 = 1 - s(x)s(x+\hat{\mu}) \rightarrow$ curvature coupling [like Maxwell $(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$]:



parallel transport:

$\frac{1}{2}$[right $\circ$ up $-$up $\circ$ right]$^2 = 1 -$
$\sigma(x,\mu)\sigma(x+\hat{\mu},\nu)\sigma(x,\nu)\sigma(x+\hat{\nu},\mu)$
$=: 1 - \sigma_p(x,\mu,\nu)$ 'plaquette'

$$-S(\sigma) = \beta \sum_{x,\mu<\nu} \sigma_p(x,\mu,\nu)$$

local gauge invariance (group $\mathrm{Z}(2)^{\#\,\mathrm{of\,sites}}$) [analog: $A_\mu \to A_\mu + \partial_\mu \alpha$]:

$$S(\sigma) = S(\sigma'), \quad \sigma'(x,\mu) = s(x)\sigma(x,\mu)s(x+\hat{\mu}), \quad s(x) = \pm 1$$

$$Z_0 = \sum_\sigma \mathrm{e}^{-S(\sigma)}, \quad \langle \mathcal{O} \rangle = \frac{1}{Z_0} \sum_\sigma \mathcal{O}(\sigma) \mathrm{e}^{-S(\sigma)}$$

- only invariant $\mathcal{O}(\sigma) = \mathcal{O}(\sigma')$ have $\langle \mathcal{O} \rangle \neq 0$ and are physical
- example: $\mathcal{O}(\sigma) = \prod_i \sigma(x_i, \mu_i)$ where links $\{(x_i, \mu_i)\}$ are a closed curve on the lattice (Wilson loop)
- special case: straight line closing by periodicity (Polyakov line)

we split: $x = (x_0, \vec{x})$ ($\vec{x}$: $D-1$ dimensional), $\pi(\vec{x}) := \prod_{x_0} \sigma_0(x)$,

$$\langle \pi(\vec{x}) \rangle = 0 \,(\text{by symmetry}), \quad \langle \pi(\vec{x}) \pi(\vec{y}) \rangle = G(\vec{x} - \vec{y})$$

- confined (disordered phase): ‘area law’: $G(\vec{x}) \propto \mathrm{e}^{-K L_0 |\vec{x}|}$
- K generalizes $1/\xi$, mass gap

Worm $\rightarrow$ jellyfish (Irukandji?)

$Z_0 = \sum_\sigma \ldots \rightarrow$ all order graph expansion strictly analogous to spin case

- expand: $e^{\beta\sigma_p} = \cosh\beta \sum_{n=0,1} (\tanh\beta\sigma_p)^n$, $n \rightarrow n(x,\mu,\nu) \equiv n_{\mu\nu}(x)$
- for each plaquette config. $\{n(x,\mu,\nu)=0,1\}$ we sum over $\sigma(x,\mu)$ $\Rightarrow$ constraint on $n(x,\mu,\nu)$, in words: at each link an even $\#$ of $n(x,\mu,\nu)=1$ must touch
- subset of $n=1$ plaquettes which form a closed surface [generalized: even branchings, disconnected components...]

$$Z_0 = \sum_n (\tanh\beta)^{\sum_{x,\mu<\nu} n_{\mu\nu}(x)} \delta[\partial^*_\mu n_{\mu\nu}]$$

$$\delta[\partial^*_\mu n_{\mu\nu}] := \prod_{x\mu} \delta_{\partial^*_\mu n_{\mu\nu},\text{even}}$$

- updates preserving constraint, cube flip (CF): $n \rightarrow 1-n$ on plaquettes forming a 3-cube (in D dim). $\rightarrow$ works, but CSD

Attempts to generalize PS idea

- $\rightarrow$ allow surfaces with defects
- which kind? smallest possible?

$$Z_j = \sum_{\sigma} \mathrm{e}^{-S(\sigma)} \prod_{x\mu} \sigma(x,\mu)^{j(x,\mu)}, \quad j(x,\mu) \in \{0,1\}$$

- symmetries $\Rightarrow$ $Z_j \neq 0$ only if:
  - $\partial^*_\mu j_\mu(x) = 0$ mod 2 [like $k(x,\mu)$ for $\mathcal{G}_0$ graphs]
  - j has zero winding number in all dirs
  - Wilson loops, pairs of Pol. lines, or very irregular networks
- c.f. spin model: global Z(2) symm. $\Rightarrow$ even number of defects
  - discrete, two = smallest nontrivial set

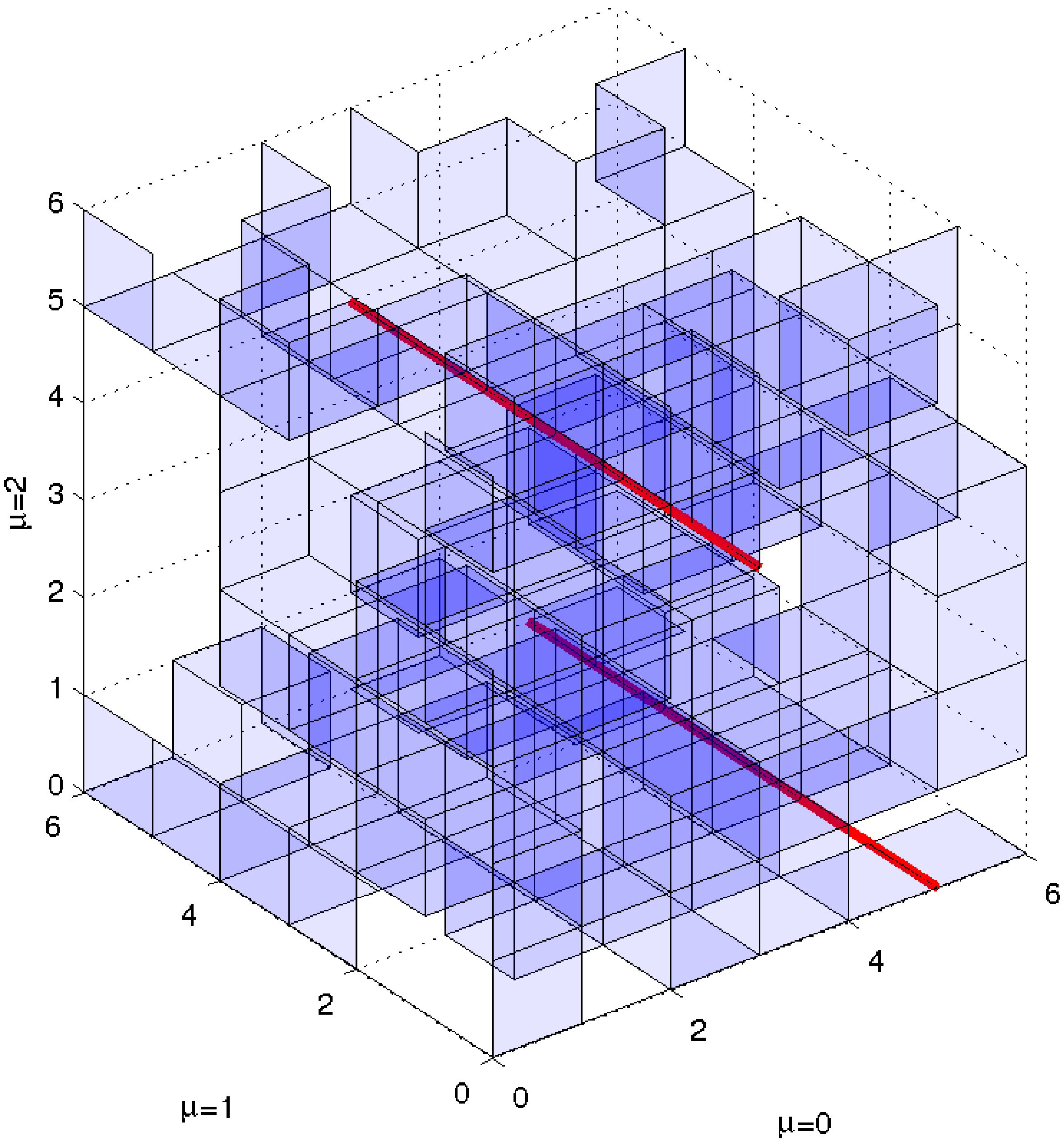
$$\mathcal{Z} = \sum_j Z_j R^{-1}(j_\mu) = \sum_n (\tanh\beta)^{\sum_{x,\mu<\nu} n_{\mu\nu}(x)} R^{-1}(\partial^*_\mu n_{\mu\nu})$$

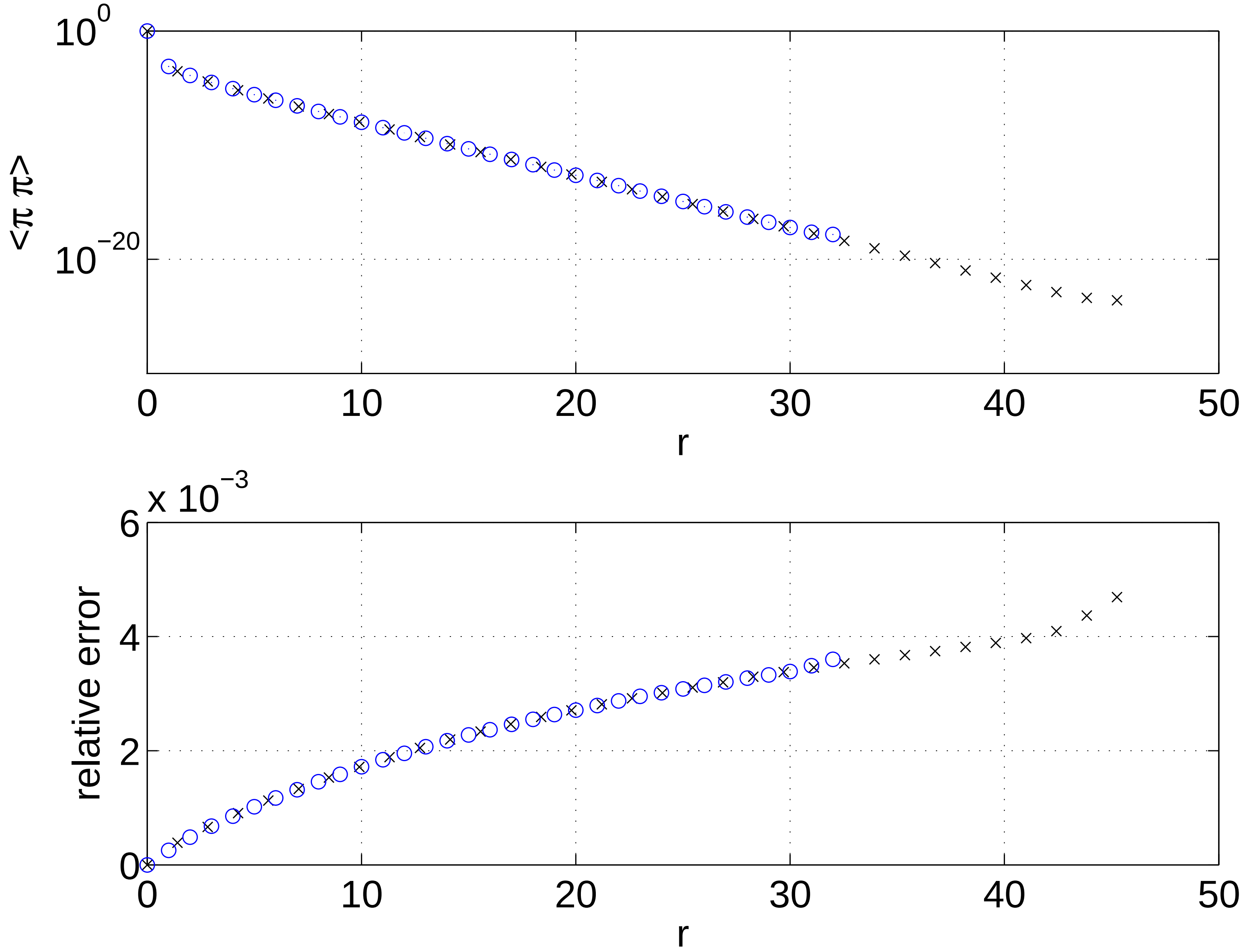
- correct algos, but we could not find an R that keeps defect set 'small' and yields efficient dynamics; one (of many) attempt

$$R^{-1}(j) = \mathrm{e}^{-\kappa \sum_{x,\mu} j(x,\mu)}$$

- $\rightarrow$ concentrate on improving $\langle \pi(\vec{x})\pi(0)\rangle$ (Polyakov)
- $j_{\vec{u},\vec{v}}$ current corresponding to two Polyakov lines
  - shift of lines + flip 'ladder' of $n(x,\mu,\nu)$
  - plus CF for ergodicity (done around defect lines)
- take $R \propto \mathrm{e}^{-\alpha|\vec{u}-\vec{v}|}$ with expected area law

same improvement of the correlator as in the Ising model


μ=2
μ=1
μ=0
0
1
2
3
4
5
6
0
2
4
6
0
2
4
6



10^0
10^−20
<π π>
0
10
20
30
40
50
r
x 10^−3
6
4
2
0
relative error
0
10
20
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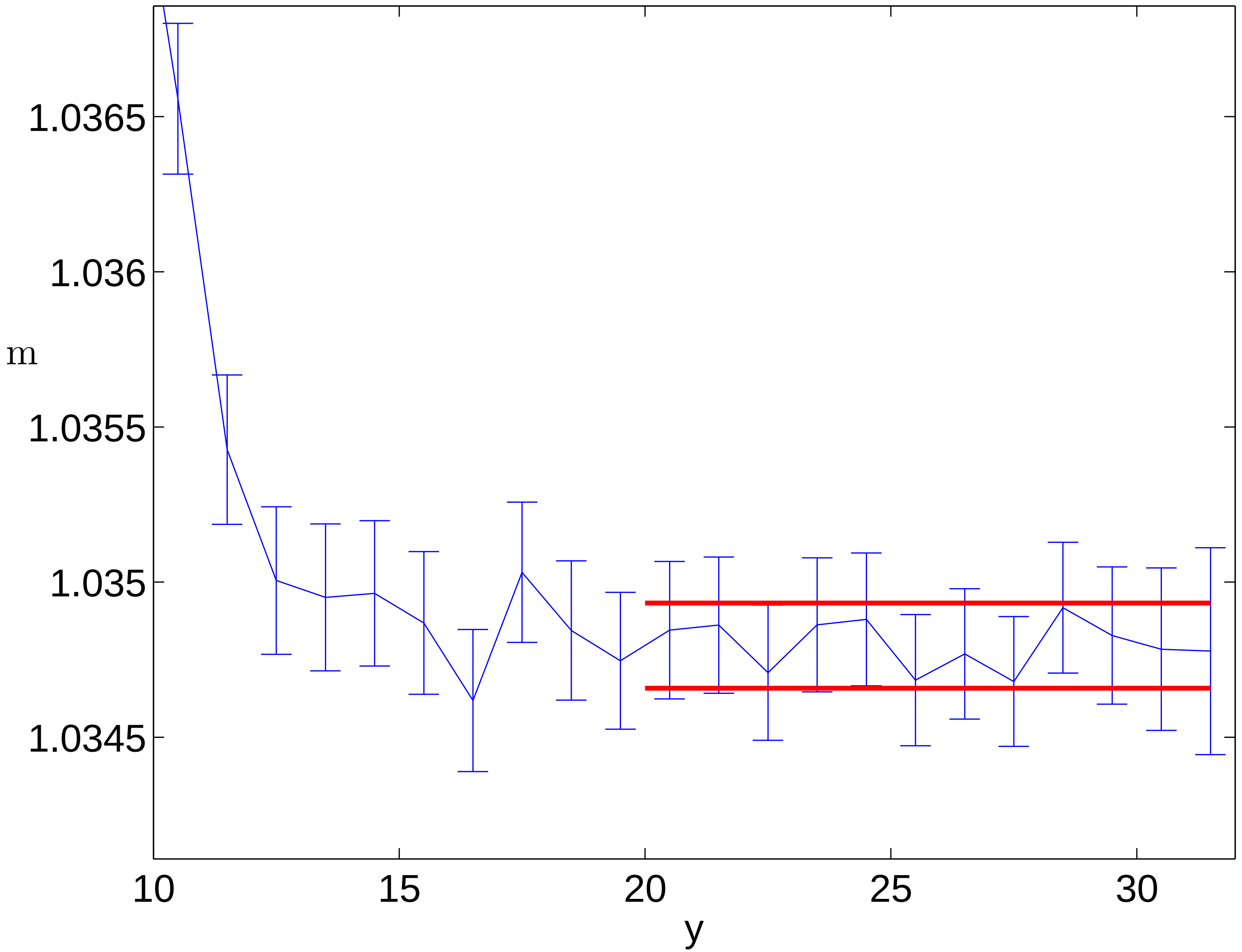
40
50
r


1.0365
1.036
1.0355
1.035
1.0345
m
10
15
20
25
30
y

Conclusions

- PS: very simple clever idea, could have been done long ago
- not covered here: successful generalizations to O(N) sigma models (N-vector model), CP(N) models and $2D$ fermions
- not just a new algorithm, but simulation of nontrivially transformed model (‘partial duality transformation’)
- merits may depend on observables of interest
- generalization to gauge models very nontrivial (as with clusters)
- reason different geometry:
  - configs: loops $\rightarrow$ surfaces
  - defects: points $\rightarrow$ loops [much ‘larger’ manifold]
- not covered here: the high precision estimates of the string tension allow for interesting checks of the low energy effective string model description of gauge theories (Symanzik, Lüscher)

did you see the jellyfish?