QCD parameters from the lattice: the ALPHA Collaboration method

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Overview:

- perturbative: PQCD
- PT = asymptotic expansion only
- Lattice: LQCD
- establish connection: PQCD=LQCD
- Schrödinger Functional, step scaling method
- Λ parameter, RGI masses, a few results

QCD: PT in a nutshell

bare QCD Lagrangian: $\quad \mathcal{L}_{\rm QCD} \equiv \mathcal{L}(A_\mu, q, \bar{q}\,; g_0, m_{0,f})$

- bare coupling g_0, bare quark masses $m_{0,f}$, $f =$ d,u,s,c,b,t $[N_f + 1$ param.]
- PT: $\mathcal{L} \rightarrow$ algorithm to generate Feynman rules
- $D = 4 - 2\varepsilon,\ g_0^2 = g_{\overline{\rm MS}}^2\,\mu^{2\varepsilon} Z_g,\ m_{0,f} = m_{\overline{\rm MS},f} Z_m$
- renormalization: eliminate $g_0, m_{0,f}$, send $\varepsilon \rightarrow 0$
- PT physical predictions in terms of μ, $g_{\overline{\rm MS}}$, $m_{\overline{\rm MS},f}$ $[N_f + 2$ param.]
- RG redundancy: there are $(\delta\mu, \delta g_{\overline{\rm MS}}, \delta m_{\overline{\rm MS},f})$ with unchanged physics $\Rightarrow$ running $\bar{g}(\mu), \overline{m}_f(\mu)$, solutions of

$$\mu\frac{\partial}{\partial\mu}\bar{g} = \beta(\bar{g}), \qquad \mu\frac{\partial}{\partial\mu}\overline{m}_f = \tau(\bar{g})\overline{m}_f \qquad [\text{all } \overline{\rm MS}]$$

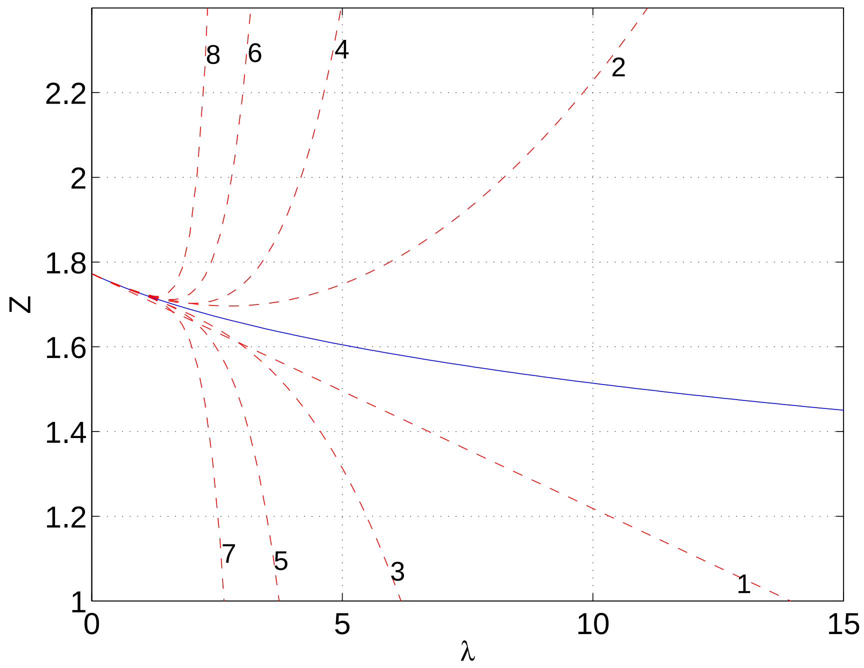
- PT results: truncated series in $\bar{g}(\mu)$, best approximation with $\mu \sim$ physics

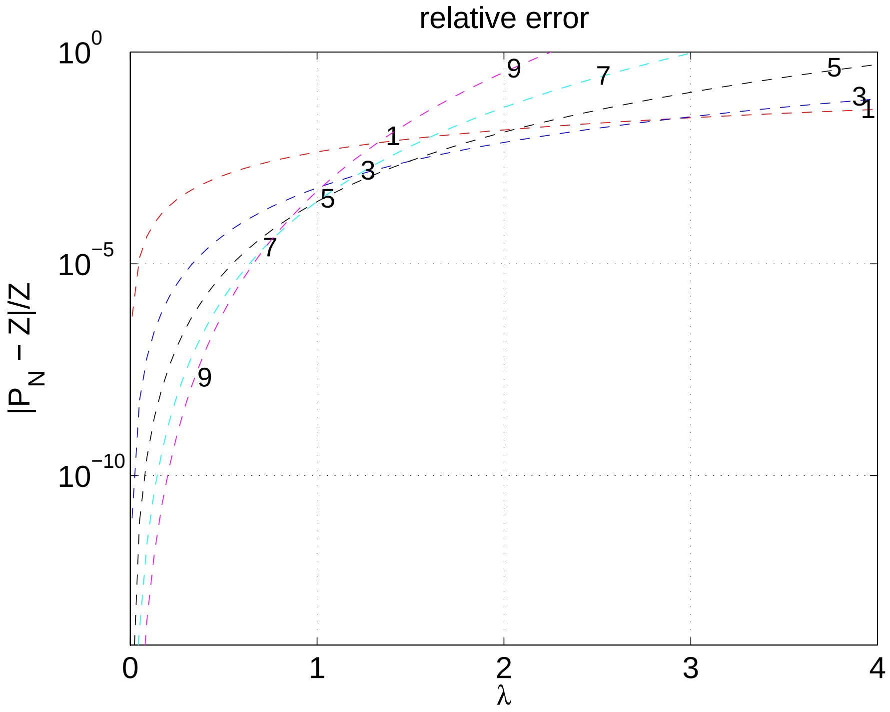
What is PT?

'convergence'? $\longrightarrow$ asymptotic series

$$\text{toy example:} \quad Z(\lambda) = \int_{-\infty}^{\infty} d\phi\, e^{-\phi^2 - \frac{\lambda}{4!}\phi^4} = \sqrt{6/\lambda}\; e^{3/\lambda} K_{1/4}(3/\lambda)$$

$$c_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} Z(\lambda)|_{\lambda=0} = \frac{\Gamma(2n+1/2)}{\Gamma(n+1)}, \qquad \left| Z(\lambda) - \sum_{n=0}^{N} c_n \lambda^n \right| \propto \lambda^{N+1} \quad \text{as} \quad \lambda \searrow 0$$





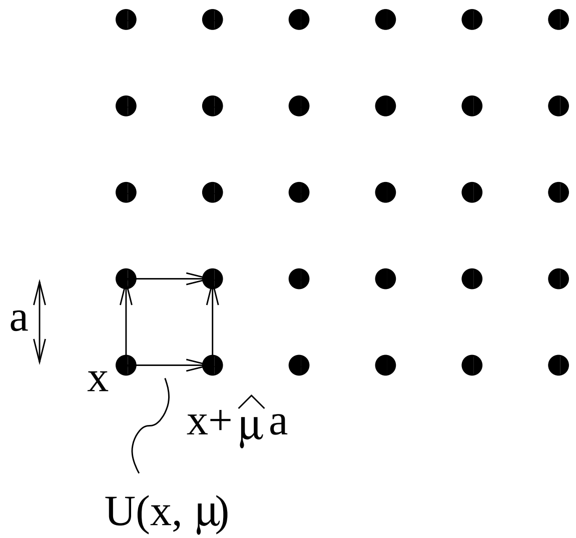
regions in λ: $\quad \mathrm{I} = [0,1]\{\mathrm{PT}\}, \quad \mathrm{II} = [1,5]\{\mathrm{PP}\}, \quad \mathrm{III} = [5,\infty]\{\mathrm{NP}\},$

Why Lattice QCD?

- to conceptually define without PT what is approximated by PT
- to allow computations beyond PT (in II, III, usually MC simulations)

$\Longrightarrow$ the path integrals of (Euclidean) QFT are approximated by sequences of well-defined finite-dimensional compact integrations (SU(3)+Grassmann).
requires:

- finite space-time resolution: lattice spacing $a > 0\,[\mathrm{fm}]$
- finite space-time volume L^4, $L < \infty[\mathrm{fm}]$
- then: #dgf $= (L/a)^4 \times$ [color,flavor,spin] $=$ finite



$\int d^4x (\partial_\mu \varphi)^2 \rightarrow a^4 \sum_{x,\mu} \frac{1}{a^2}[\varphi(x + a\hat{\mu}) - \varphi(x)]^2$,
$\int d^4x\, F_{\mu\nu}^2 \rightarrow a^4 \sum_{x,\mu<\nu}$ (plaquette terms), etc.

criteria:

- naive (classical) continuum limit
- renormalizability: complete set of local terms up to dimension 4 compatible with the (lattice) symmetries

Scales in (L)QCD

- dimensionless bare input parameters (‘control knobs’): $g_0, am_{0,f}, L/a$
- ‘traditional’ simulations: baryon masses, matrix elements (same order GeV, extracted from correlations) $\longrightarrow$ tune knobs such that:
  - continuum limit: $aM_{\rm had} \ll 1, aF_{\pi,K} \ll 1, \ldots.$ simultaneously
  - arrange $Lm_\pi \gtrsim 4$ and tune N_f inputs $\rightarrow$ rest: predicitions
  - regularization scales (a, L) remote from physical scales
  - limit (continuum): $g_0, am_{0,f}$ assume critical values, $L/a \rightarrow \infty$
  - [in practice (2011): $\ll 1 \leftrightarrow < 1/3$, $L/a \lesssim 100$, $a > 0.05\,{\rm fm}$]
  - $M'_{\rm had}/M_{\rm had}$ etc. become independent of the lattice = have been NP computed
  - systematic errors (‘untaken’ limits): $\mathrm{O}(a^2 M_{\rm had}^2)$ and $\exp(-cLM_{\rm had})$
- continuum limit works at finite $LM_{\rm had}$ (even small, ‘Femto’ universe)
  $\Rightarrow$ universal results which depend on (e.g. periodic) boundary conditions
  L^{-1} somewhat like an external momentum probe
- for $LM_{\rm had} \ll 1$ we may use PT $[\bar{g} \propto -1/\ln(L\Lambda)]$ again (lattice or other)

LQCD=PQCD=QCD

PQCD and LQCD almost look like separate theories, but in principle we are able to compute 'connecting' dimensionless numbers:

$$\bar{g}\,(\mu), \frac{\overline{m}_f(\mu)}{F_K} \quad \text{at} \quad \mu = 100 F_K$$

Problems with this:

- NP calculation of PT parameters
- multiscale: UV $\rightarrow a^{-1} \gg \mu \gg F_K \gg L^{-1} \leftarrow$ IR $\qquad [L/a \lesssim 100]$

Solution of the ALPHA collaboration:

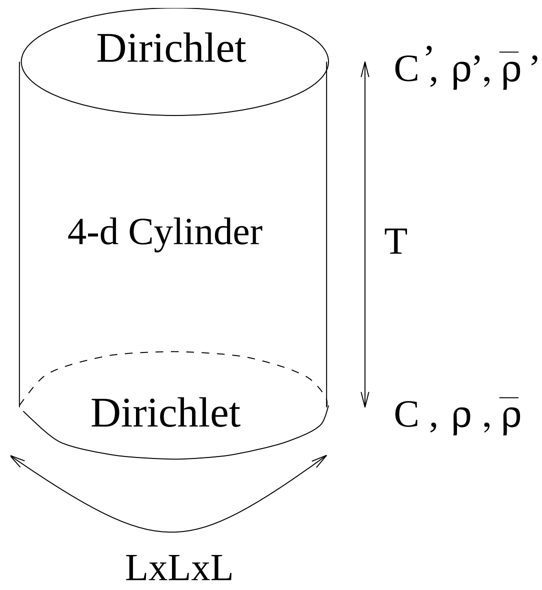
- intermediate use of a Femto universe with $\mu = L^{-1}$ to:
- define an NP scheme with $\bar{g}_F(L)$ and $\overline{m}_{F,f}(L)$ $\quad$ [before: $\bar{g} \equiv \bar{g}_{\overline{\mathrm{MS}}}$,..]
- NP-compute evolution with L $[\rightarrow \beta_F(.), \tau_F(.)]$ at arbitrary L^{-1}
- verify agreement with PT for $L^{-1} \gg 1\,\mathrm{GeV}$
- evolve from $\bar{g}_F(L = F_K^{-1})$ up to $\bar{g}_F(L_{\min})$, $L_{\min}^{-1} \gg 1\mathrm{GeV}$
- convert by PT (controlled, region I) $\bar{g}_F(L_{\min})$ to $g_{\overline{\mathrm{MS}}}(L_{\min}^{-1})$
- problem solved (take $F_K L_{\min} = 0.01$), no more reference to finite volume

Schrödinger Functional (SF) scheme

$$\langle C', \rho', \bar\rho\,'|\mathrm{e}^{-T\hat H}P_g|C, \rho, \bar\rho\,\rangle$$

$$= \int_{\mathrm{Dirichlet}\; b.c.} D[\mathrm{fields}]\,\mathrm{e}^{-S}$$

$$= \mathrm{e}^{-\Gamma}$$



$T = L$ usually

$\mathrm{e}^{-a\hat H}$	:	transfer matrix
P_g	:	projector on gauge invariant states
$\lvert C, \rho, \bar\rho\,\rangle$	:	field diagonal quantum state
C, C'	:	spatial gauge fields (constant diagonal, parameter η)
$\rho', \bar\rho\,', \rho, \bar\rho$	:	boundary quark fields (Grassmann source)

$$\frac{\partial\Gamma}{\partial\eta} = \frac{k}{\bar g^2_{\mathrm{SF}}(L)}, \qquad k \leftrightarrow \bar g^2_{\mathrm{SF}} = g_0^2 + \ldots.$$

$\Leftrightarrow$ response in free energy to deforming b.c. $\leftrightarrow$ Casimir effect design criteria were:

- good signal in Monte Carlo
- manageable PT (2 loops) to connect to $\overline{\mathrm{MS}}$ scheme
- small cutoff effects, $\mathrm{O}(a)$ can be canceled (Symanzik improvement)
- IR finite at finite L, T, thus $\overline{m}_f = 0$ possible $\rightarrow$
- massless scheme [Wilson lattice fermions: $am_{0,f} = am_c(g_0)$]

Step scaling function

like β function, but $\mu = L^{-1} \rightarrow \mu/2$ instead of $\mu \rightarrow \mu + \delta\mu$

$$\Sigma(u, a/L) = \bar{g}^2_{\mathrm{SF}}(2L)|_{\bar{g}^2_{\mathrm{SF}}(L)=u}$$

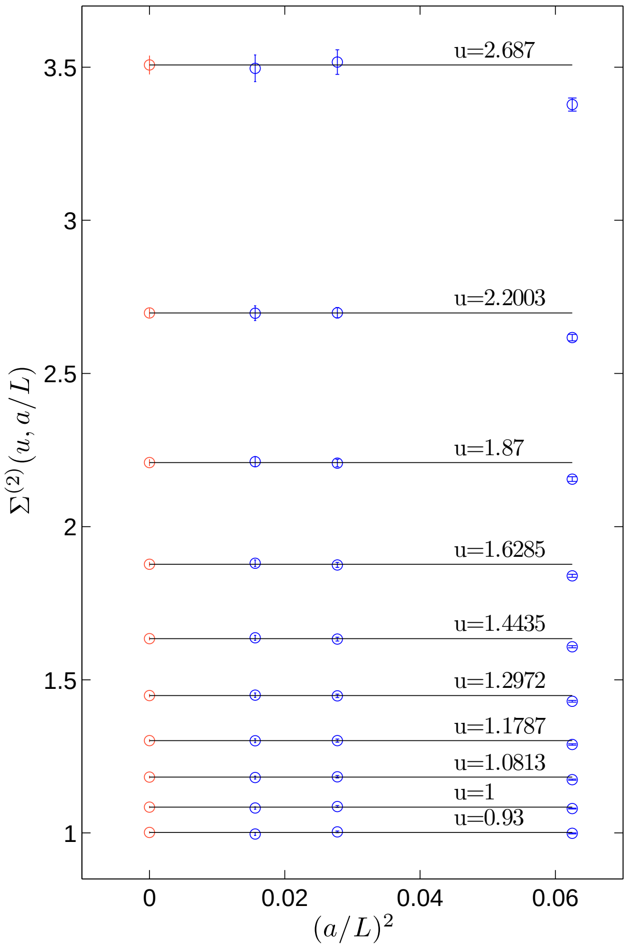
operationally [in principle..]:

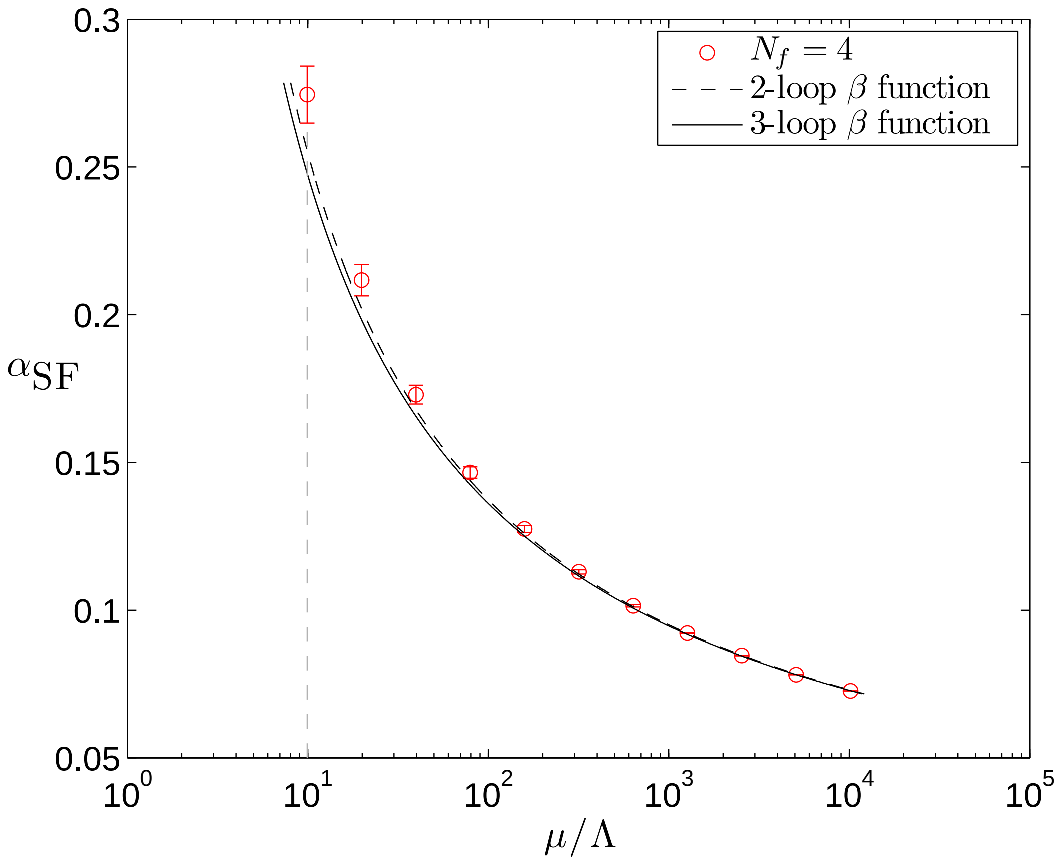
- choose L/a, set $m_{0,f} = m_c(g_0)$ [predetermined]
- tune g_0 to achieve $\bar{g}^2_{\mathrm{SF}}(L) = u$, then $L/a \rightarrow 2L/a$, return $\Sigma = \bar{g}^2_{\mathrm{SF}}(2L)$
- repeat for a number of $L/a = 6, 8, 10, \ldots$ to take the continuum limit

$$\sigma(u) = \lim_{a/L \rightarrow 0} \Sigma(u, a/L) \qquad \Rightarrow -\frac{1}{2}\int_u^{\sigma(u)} \frac{dx}{\sqrt{x}\,\beta_{\mathrm{SF}}(\sqrt{x})} = \ln 2$$

- scales: $a \ll L, 2L$ only

Some recent results





PhD, Fatih Tekin, 2010, $\alpha_{\rm SF} = g^2_{\overline{\rm MS}}/4\pi$

Λ parameter and RGI masses

$$\mu \frac{\partial}{\partial \mu} \bar{g} = \beta(\bar{g})$$

$$\to \Lambda = \mu (b_0 \bar{g}^2)^{-b_1/2b_0^2} \mathrm{e}^{-1/2b_0 \bar{g}^2} \exp\left\{ -\int_0^{\bar{g}} dx \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x} \right] \right\} \quad [\bar{g} \equiv \bar{g}(\mu)]$$

- $d\Lambda / d\mu = 0$ due to RG eq.
- Λ is the integration constant and selects a definite trajectory $\bar{g}(\mu)$
- for large μ a PT approximation of β is sufficient
- knowledge: large μ asymptotics of $\bar{g}(\mu) \Leftrightarrow \Lambda$ [given $\beta_{3\mathrm{loop}}$]
- $\Lambda_{\mathrm{scheme1}} / \Lambda_{\mathrm{scheme2}}$ exact in 1-loop PT (compute for $\mu \to \infty$)

From $\mu \frac{\partial}{\partial \mu} \overline{m}_f = \tau(\bar{g}) \overline{m}_f$ by a similar construction $\to M_f^{\mathrm{RGI}}$ (scheme independent)

$$M_f^{\mathrm{RGI}} = \overline{m}_f (2 b_0 \bar{g}^2)^{-d_0/2b_0} \exp\left\{ -\int_0^{\bar{g}} dx \left[\frac{\tau(x)}{\beta(x)} - \frac{d_0}{b_0 x} \right] \right\} \quad [\bar{g} \equiv \bar{g}(\mu), \overline{m} \equiv \overline{m}(\mu)]$$

Recursive computation of $\Lambda_{\rm QCD}$

- compute SSF $\sigma(u)$ [at $N_f = (0,2), 4$] for a set $\{u_i\}$ at a certain error level
- interpolate σ for an interval $u \in [0, u_{\rm max}]$ with negligible extra error

Start by defining a scale $L_{\rm max}$ in the hadronic range $[L_{\rm max} = {\rm O}(F_K^{-1})]$ by

$$\bar{g}^2_{\rm SF}(L_{\rm max}) = u_{\rm max} \quad (\text{example: } u_{\rm max} = 3.45,\, N_f = 4)$$

- compute $\bar{g}^2_{\rm SF}(L_k), \quad L_k = 2^{-k} L_{\rm max}$ by using σ [this is in the continuum!]
- at $k \sim 10$ the running of $\bar{g}^2_{\rm SF}(L_k)$ is demonstrably PT
- compute $\Lambda_{\rm SF}$ from $\bar{g}^2_{\rm SF}(L_{10})$ by PT [in terms of $2^{10} L_{\rm max}^{-1}$!]
- convert $\Lambda_{\rm SF}$ to $\Lambda_{\overline{\rm MS}}$
- error analysis: propagate the errors of the sampling points (Stützstellen) of σ through the complete procedure

Finish by computing ('scale setting', trade $L_{\max} \to F_K$):

- $F_K L_{\max} = $ dimensionless universal number
- sequence of lattice(s) with: $a \searrow 0$, principle operational sequence of steps:
  - pick $L/a = 8$, tune g_0 such that $\bar{g}_{\mathrm{SF}} = 3.45$ with $m_{0,f} = m_c(g_0)$
  - thus here: $L_{\max} = 8a$
  - same[1] g_0 (hence a), choose L'/a larger (40?) and fix $am_{0,f}$ by $m_{\mathrm{meson},f}/F_K = $ experiment (physical quark masses)
  - output: aF_K (on $L' \approx \infty$) and thus $F_K L_{\max} = (aF_K)\underbrace{(L_{\max}/a)}_{=8}$
  - verify $L'F_K$, $L'M_{\mathrm{had}}$ was large enough
  - extrapolate this: $L/a = 8, 12, 12, 16, \ldots, \infty$
  - scales: $a \ll F_K^{-1}, L_{\max} \ll L'$

1. simplified, Symanzik improvement: $g_0 \to \tilde{g}_0$

Numerical example $\Lambda_{\mathrm{SF}} L_{\mathrm{max}}$ @ $N_f = 4$

[Data: Rainer Sommer, α_s workshop, Munich 2011]

$\Longrightarrow \bar{g}^2_{\mathrm{SF}}(L_{\mathrm{max}}) = u_{\mathrm{max}} = 3.45$ defines L_{max}

Fit I (gobal) — Fit II ($8 \rightarrow 16$ only)

i	$u_i = \bar{g}^2_{\mathrm{SF}}$	$\ln(\Lambda L_{\mathrm{max}})_{\mathrm{3loop}}$	$u_i = \bar{g}^2_{\mathrm{SF}}$	$\ln(\Lambda L_{\mathrm{max}})_{\mathrm{3loop}}$
0	3.45	− 2.028	3.45	− 2.028
1	2.666(46)	− 2.066(56)	2.660(21)	− 2.073(26)
2	2.179(45)	− 2.105(83)	2.173(20)	− 2.116(37)
3	1.847(37)	− 2.141(97)	1.842(17)	− 2.153(44)
4	1.606(30)	− 2.17(10)	1.602(14)	− 2.185(50)
5	1.422(25)	− 2.20(11)	1.419(13)	− 2.213(56)
6	1.278(20)	− 2.23(11)	1.275(11)	− 2.238(63)
7	1.161(17)	− 2.25(12)	1.159(10)	− 2.259(70)
8	1.064(15)	− 2.27(12)	1.0626(95)	− 2.278(76)
9	0.982(13)	− 2.29(12)	**0.9815(87)**	**− 2.294(83)**
10	0.913(11)	− 2.30(12)	0.9122(81)	− 2.309(89)

Scale setting: so far only in terms of r_0 and for $N_f = 2$ ($\rightarrow$ future plans)

(also from Rainer) Status of $r_0\Lambda_{\overline{\mathrm{MS}}}$, $r_0 = 0.475$ (?)fm

$N_f =$	0	2	4	5
ALPHA	0.60(5)	0.73(3)(5)	0.XX(5)*	
Bethke			0.71(3)	0.52(2)
Blümlein et al.			0.54(8)	

∗ in a few years

Rainers master formula:

$$\frac{\Lambda^{(5)}_{\overline{\mathrm{MS}}}}{F_K} = \underbrace{\frac{1}{F_K L_{\mathrm{max}}}}_{\mathrm{NP}} \times \underbrace{\frac{L_{\mathrm{max}}}{L_k}}_{=2^k\,[\leftrightarrow \bar{g}_{\mathrm{SF}}(L_k)]} \times \underbrace{\Lambda^{(4)}_{\mathrm{SF}} L_k \times \frac{\Lambda^{(4)}_{\overline{\mathrm{MS}}}}{\Lambda^{(4)}_{\mathrm{SF}}} \times \frac{\Lambda^{(5)}_{\overline{\mathrm{MS}}}}{\Lambda^{(4)}_{\overline{\mathrm{MS}}}}}_{\mathrm{PT}}$$

Conclusions

- arduous long term project (worthwhile? ... up to you to judge)
- stream of technical spin-offs [finite size schemes, NP renormalization, NP improvement, finite lattice PT (A3!), MC technology, ...] has influenced the field
- recent application: conformal window, walking technicolor
- short cut approaches (single lattice, a^{-1} as PT scale) have less systematic error control
- method extends to quark masses (# for $r_0 M_s^{\mathrm{RGI}}$ at $N_f = 2$ under revision)

People

[Martin Lüscher, Peter Weisz], Rainer Sommer, U.W.
Michele Della Morte, Francesco Knechtli, Juri Rolf, Roland Hoffmann, Andreas Jüttner, Björn Leder, Oliver Witzel, Harvey Meyer, Fatih Tekin, Shinji Takeda, Hubert Simma, Stefan Schäfer, Marina Marinkovic, Patrick Fritzsch