Worm algorithms: from loops to surfaces, from spin models to gauge theories

Sign2014 Darmstadt, Feb. 18, 2014

by Ulli Wolff

Humboldt Universität, Berlin

Spin models as loop gases

$\Rightarrow$Principle demonstrated here for Ising on a D dimensional lattice

general correlation:

$$\langle \prod_x \sigma(x)^{q(x)} \rangle = \frac{Z[q]}{Z[0]}, \quad \sigma\text{: Ising spin} \pm 1, \qquad q(x) \in \{0,1\}$$

example:

$$\langle \sigma(u)\sigma(v) \rangle \qquad \leftrightarrow \qquad q = q_{u,v}, \quad q_{u,v}(x) = \delta_{x,u} + \delta_{x,v}$$

'partition function with charges (point defects)':

$$Z[q] = \sum_\sigma \mathrm{e}^{\beta \sum_{x,\mu} \sigma(x)\sigma(x+\hat{\mu})} \prod_x \sigma(x)^{q(x)}$$

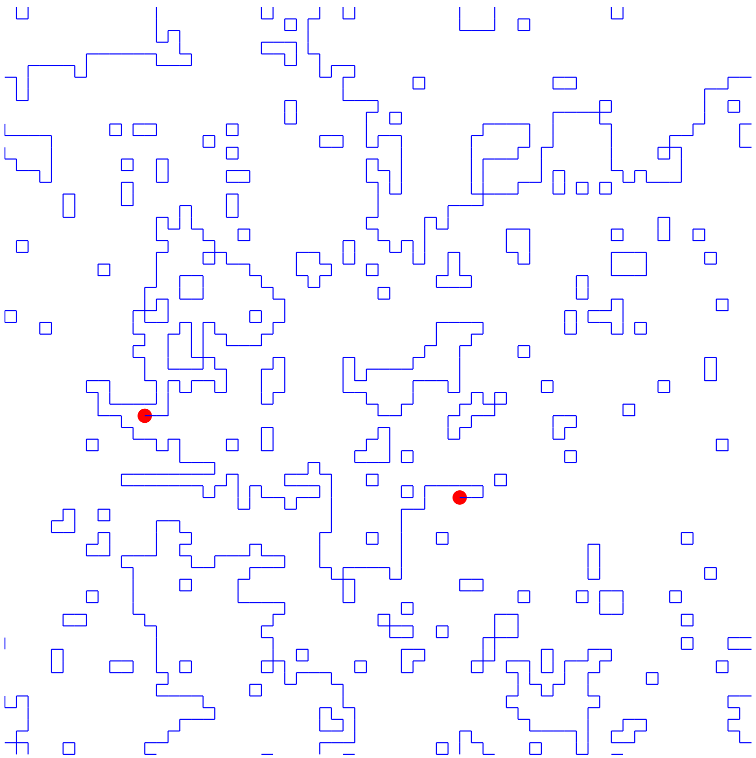
- global Z(2) symmetry $\Rightarrow$ $Z[q] = 0$ unless $\sum_x q(x) = \text{even}$ (constraint)
- use: $\mathrm{e}^{\beta\sigma(x)\sigma(x+\hat{\mu})} = c \sum_{k=0,1} t^k [\sigma(x)\sigma(x+\hat{\mu})]^k, \quad c = \cosh\beta, t = \tanh\beta$
- insert on each link $\Rightarrow \{k(x,\mu) \equiv k_\mu(x) = 0,1\}$, then sum-out σ

(dropping factor $c^{\#\text{links}}$):

$$Z[q] = \sum_k t^{\sum_{x,\mu} k(x,\mu)} \, \delta[\partial^*_\mu k_\mu - q]$$

- constraint $\delta[\ldots]$: $\sum_\mu \left(k(x,\mu) - k(x-\hat{\mu},\mu)\right) \equiv q(x) \pmod 2$

→loop graphs, example with $q_{u,v}$:



defect ensemble:

$$\mathcal{Z} = \sum_q R[q] Z[q] = \sum_{k,q} R[q]\, t^{\sum_{x,\mu} k(x,\mu)}\, \delta[\partial^*_\mu k_\mu - q]$$

- a priori free choice of $R[q] \geqslant 0$ to define $\mathcal{Z}$ ensemble $\rightarrow \langle\langle \ldots \rangle\rangle$
- only even q appear ($\leftrightarrow$original symmetry)

observables now by counting:

$$\langle \sigma(u)\sigma(v)\rangle = \frac{\langle\langle \delta[\partial^*_\mu k_\mu - q_{u,v}]\rangle\rangle}{\langle\langle \delta[\partial^*_\mu k_\mu]\rangle\rangle} \times \frac{R[0]}{R[q_{u,v}]}$$

or

$$\frac{\left\langle\left\langle \delta[\partial^*_\mu k_\mu] \sum_{x,\mu} k(x,\mu)\right\rangle\right\rangle}{\langle\langle \delta[\partial^*_\mu k_\mu]\rangle\rangle} \leftrightarrow \text{internal energy}$$

side-remark: duality transformation$\leftrightarrow$
solve constraint $\partial^*_\mu k_\mu = 0$: $k_\mu = \varepsilon_{\mu\nu\lambda_1\ldots\lambda_{D-2}} \partial^*_\nu \sigma_{\lambda_1\ldots\lambda_{D-2}}$ with 'free' σ
σ: spin in $D = 2$ (self-dual), gauge in $D = 3$,...
This step is missing for the loop ensemble....

- disjoint classes of q, $\sum_x q(x) = 0, 2, 4, 6, \ldots$
- A: restrict $(R > 0)$ to class 0:
  - 'vacuum' graphs only
  - local updates (flip $k(x, \mu)$) around plaquettes
  - critical slowing down, few observables accessible
- B: allow 0 and 2 ($q \equiv q_{u,v}$ with arbitrary u, v; 0 for $u = v$)
  - local moves of u or v (k 'follows' due to constraint)
  - almost no csd $\leftarrow$ excursion to large phase space advantageous!
    $\rightarrow$Prokof'ev and Svistunov, 2001
  - 2-point function naturally accessible
  - 'perfect' estimator with $R[q_{u,v}] = \rho^{-1}(u - v)$
    favoring large separations

generalizations proven: Potts, XY, O(N) nonlinear σ-models, CP(N) models
not yet: SU(N) spins (SU(N)$\times$SU(N) invariant principle chiral models)

Gauge models as surface gases

$\Rightarrow$Principle demo here for Z(2) gauge model on a D dimensional lattice
[expect $\to$ U(1) obvious, non-Abelian, SU(3): another story...]
general correlation of gauge field $\sigma(x,\mu) = \pm 1$:

$$\langle \prod_{x\mu} \sigma(x,\mu)^{j(x,\mu)} \rangle = \frac{Z[j]}{Z[0]}, \quad \sigma(x,\mu)\text{: Ising gauge} \pm 1, \qquad \text{flux } j(x,\mu) \in \{0,1\}$$

example, Wilson loop

$$\langle \sigma[\gamma] \rangle \qquad \leftrightarrow \qquad j = j^{(\gamma)}, \quad j^{(\gamma)}(x,\mu) = \sum_{y\nu \in \gamma} \delta_{x\mu,y\nu}$$

'partition function with currents (line defects)':

$$Z[j] = \sum_{\sigma} \mathrm{e}^{\beta \sum_p \sigma_p(x)} \prod_{x\mu} \sigma(x,\mu)^{j(x,\mu)} \qquad p \leftrightarrow \text{plaquettes}$$

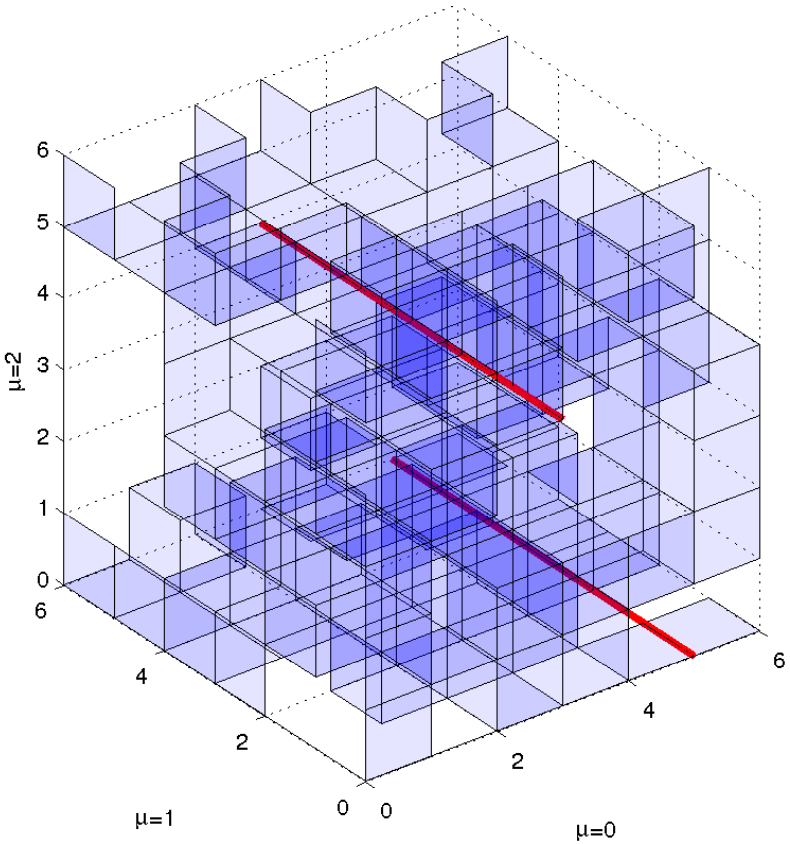
- local Z(2) symmetry $\Rightarrow$ $Z[j] = 0$ unless $\partial^*_\mu j_\mu(x) = 0 \pmod 2$
- center symm. on torus $\Rightarrow$ $Z[j] = 0$ unless $\sum_{x, x_\mu \,\text{fixed}} j_\mu(x) = \text{even}$
- use: $\mathrm{e}^{\beta \sigma_p} = c \sum_{k=0,1} t^k [\sigma(l_1)\sigma(l_2)\sigma(l_3)\sigma(l_4)]^k, \quad c = \cosh\beta, t = \tanh\beta$
- insert on each plaq. $\Rightarrow \{k(x;\mu,\nu) \equiv k_{\mu\nu}(x) = 0, 1\}$, then sum-out σ

(dropping factor $c^{\#\mathrm{plaqs}}$):

$$Z[j] = \sum_k t^{\sum_{x,\mu} k(x;\mu,\nu)} \, \delta[\partial^*_\mu k_{\mu\nu} - j_\nu]$$

- constraint $\delta[\ldots]$: $\sum_\mu (k(x;\mu,\nu) - k(x-\hat{\mu};\mu,\nu)) \equiv j(x,\nu) \pmod 2$

$\rightarrow$surface graphs, 3D example with $j \leftrightarrow 2$ Polyakov lines:



defect ensemble:

$$\mathcal{Z} = \sum_j R[j] Z[j] = \sum_{k,j} R[j]\, t^{\sum_{x,\mu,\nu} k(x;\mu,\nu)}\, \delta[\partial^*_\mu k_{\mu\nu} - j_\nu]$$

now the analogy fades a bit....
which j do we want to allow (choice of R)??

- allowed: $j(x,\mu)$ divergence free (e.g. closed loop) + center symmetry (no odd winding number)
- !: include small defect networks, returning to vacuum sometimes

unsuccessful experiments with regard to csd in $D=3$ at β_c:

- $R[j] \propto \mathrm{e}^{-\alpha \sum_{x,\mu} j(x,\mu)} \leftrightarrow$ chemical potential to control defects
- either: single (self-avoiding) closed loop, or: arbitrary allowed j
- legal algos, correct, α does the job, but NOT FAST at β_c
- there are a lot more loop than point defects

moreover:

- ensemble of irregular fuzzy loop(s):
  can they be connected to 'useful' observables?
- e.g. string tension from fluctuating Wilson loops (not rectangular)?

Perfect Polyakov line correlation

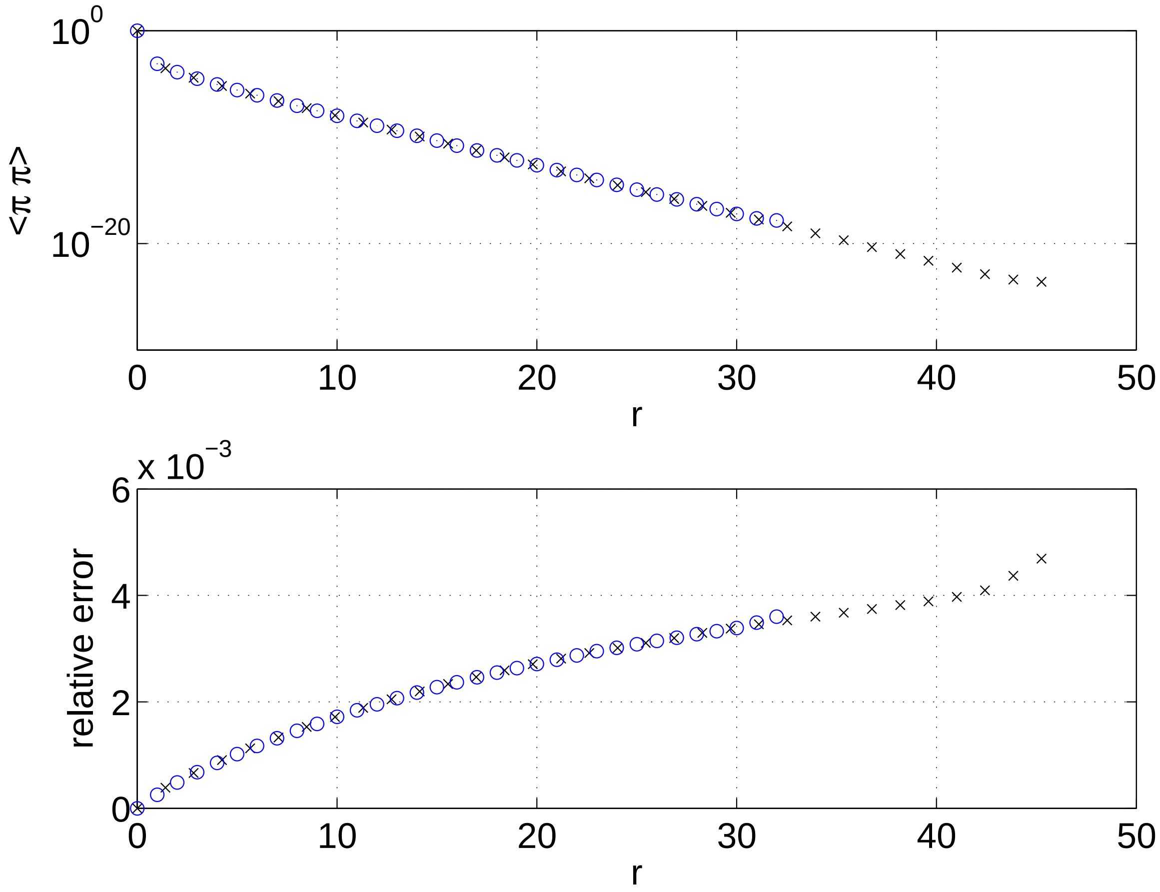
- csd cannot be overcome at the moment ($z \approx 2$)
- but improved observables available in

$$\mathcal{Z} = \sum_{\vec{u}\,,\vec{v}\,,k} R(\vec{u} - \vec{v}\,)\, t^{\sum_{x\,,\,\mu\,,\,\nu} k(x;\mu,\nu)}\, \delta\Big[\partial^*_\mu k_{\mu\nu} - j_\nu^{(\vec{u}\,,\vec{v}\,)}\Big]$$

- $j^{(\vec{u}\,,\vec{v}\,)}$ defect $\leftrightarrow$ 2 lines in 0-direction at $\vec{u}$ and $\vec{v}$ in $(D-1)$-space

$$\langle \pi(\vec{x}\,)\pi\big(\vec{0}\big)\rangle = R^{-1}(\vec{x}\,)\frac{\langle\langle \delta_{\vec{x}\,,\vec{u}\,-\vec{v}}\,\rangle\rangle}{\langle\langle \delta_{\vec{u}\,,\vec{v}}\,\rangle\rangle}$$

- update steps amalgam of:
  - flip $k(x;\mu,\nu) \to 1 - k(x;\mu,\nu)$ around 3-cubes, good acceptance
  - shift lines: propose $\vec{u} \to \vec{u} \pm \hat{i}$ & flip ‘ladder’ of plaquettes
  - ok acceptance up to 64^3 (β_c) and 48×64^2 ($\beta < \beta_c$)
  - non-rejecting shifts: different ensemble $\mathcal{Z}' \neq \mathcal{Z}$
    weight of vacuum graphs unchanged
- line-shift step can change wrapping number of surfaces
  $\leftrightarrow Z_{\text{twisted bc}}/Z = \langle\langle \ldots \rangle\rangle$ well measurable



- $\rightarrow$opportunity to precision match gauge theory $\leftrightarrow$ effective string model
- details: see my lat13 talk

A few conclusions

- worm or strong coupling graph simulation method
  - straight-forward to generalize spins → abelian gauge model
  - not so efficient for CSD (do we miss the essential trick?)
  - line defects different story from point defects....
- taylor ensemble to observables
  - this does generalize successfully
  - Polyakov loop correlation decay traced over many oders
- fermions: no news, see sign 2012