The Higgs-Yukawa Model On The Lattice

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30. September 2009

Graduiertenkolleg "Masse, Spektrum, Symmetrie"

The purpose to consider Higgs-Yukawa models

- the matter content of the standard model:

Quarks 3 colours	u (2.4 MeV)	c (1.27 GeV)	t (171.2 GeV)
	d (4.8 MeV)	s (104 MeV)	b (4.2 GeV)
Leptons	ν_e ($<$ 2.2 eV)	ν_μ ($<$ 0.17 MeV)	ν_τ ($<$ 15.5 MeV)
	e (0.511 MeV)	μ (105.7 MeV)	τ (1.777 GeV)
	1	2	3

- The symmetry: $SU(2)_W \times U(1)_Y \times SU(3)_c$

The standard model Lagrangian:

$$\mathcal{L}_{SM} = -\frac{1}{4}\underbrace{F^i_{\mu\nu}F_i^{\mu\nu}}_{SU(2)} - \frac{1}{4}\underbrace{B_{\mu\nu}B^{\mu\nu}}_{U(1)_Y} - \frac{1}{4}\underbrace{G^j_{\mu\nu}G_j^{\mu\nu}}_{SU(3)}$$

$$+\bar{\psi}_a \not{D}\, \psi_a + \frac{1}{2}(D_\mu\Phi)^\dagger(D^\mu\Phi) + \frac{1}{2}\mu^2\left(\Phi^\dagger\Phi\right) - \lambda\left(\Phi^\dagger\Phi\right)^2$$

$$-y_b \underbrace{\begin{pmatrix} \bar{t}_L \\ \bar{b}_L \end{pmatrix} \cdot \Phi\, b_R}_{\bar{b}_L\phi_0 b_R} - y_t \underbrace{\begin{pmatrix} \bar{t}_L \\ \bar{b}_L \end{pmatrix} \cdot \tilde{\Phi}\, t_R}_{\bar{t}_L\phi_0 t_R} + h.c.$$

+further Yukawa couplings

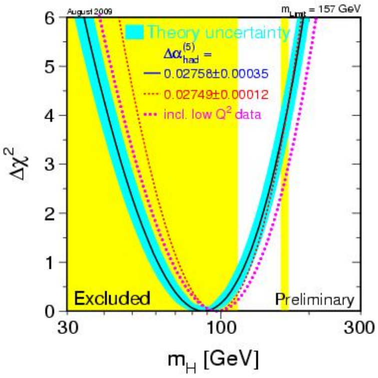
e.g. $$D_\mu\psi_e = \left(\partial_\mu - i\frac{g}{2}\vec{\tau}\cdot\vec{A}_\mu + i\frac{g'}{2}B_\mu\right)\psi_e$$

- Particle masses in the standard model (SM) are generated via couplings to the scalar Higgs doublet.
- The symmetric and the spontaneously broken phase are distinguished by the Higgs vacuum expectation value (vev)

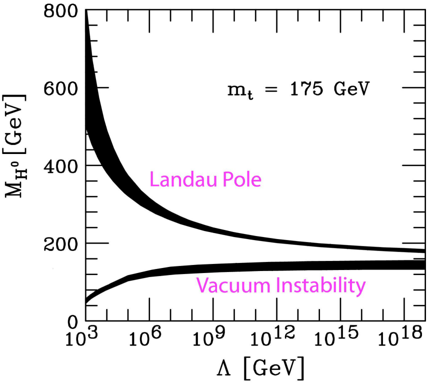
$$\langle \phi_0 \rangle = vev$$

- The phase structure has been analyzed both analytically and numerically [P. Gerhold and K. Jansen: arXiv:0705.2539, arXiv:0707.3849].

Mass bounds



http://lepewwg.web.cern.ch/LEPEWWG/



Picture from Hagiwara et. al. (PDG), Phys. rev. D66 (2002).

Purpose of lattice computations

The purpose to consider Higgs-Yukawa models on the lattice is

- to determine non-perturbatively the phase structure,
- to establish lower and upper bounds of the Higgs boson mass from first principles calculations,
- to investigate the unstable nature of the Higgs boson and compute its decay width as well as its resonance mass.

Restrictions:

- The $SU(2)$ gauge couplings g, g' are zero, such that the model possess a global $SU(2)_L \times U(1)_Y$ symmetry.
- We only consider the particles with the highest Yukawa couplings to the Higgs fiels (t, b).
- The involved heavy quarks (t, b) are considered as degenerate and so $y_t \equiv y_b$.
- We neglect the $SU(3)$ gauge fields.

The lattice action

The purely bosonic part of the action can be translated to lattice coordinates by substituting the derivatives by a finite difference

$$S_\Phi = -\kappa\, a^4 \sum_{x\in\mathbb{N}_L^4} \left\{ \Phi_x^\dagger \sum_\mu \left(\Phi_{x+\mu} + \Phi_{x-\mu}\right) + \Phi_x^\dagger \Phi_x + \hat{\lambda} \left(\Phi_x^\dagger \Phi_x - N_f\right)^2 \right\}$$

The continuum notation is recovered by inserting the definitions

$$\varphi_x =: \sqrt{2\kappa} \begin{pmatrix} \Phi_x^2 + i\Phi_x^1 \\ \Phi_x^0 + i\Phi_x^3 \end{pmatrix}, \lambda = \frac{\hat{\lambda}}{4\kappa^2}, m_0^2 = \frac{1 - 2N_f\hat{\lambda} - 8\kappa}{\kappa}$$

- In the limit $\lambda \to 0$

$$\kappa = \frac{1}{m_0^2 + 8}$$

- Symmetric phase corresponds to

$$\kappa < \kappa_c = \frac{1}{8}$$

- Broken phase corresponds to

$$\kappa_c > \kappa_c$$

The Ginsparg-Wilson relation

- Nielson-Ninomiya-theorem: It is impossible to formulate a lattice fermion action which
  1. obeys chiral symmetry: $D\gamma_5 + \gamma_5 D = 0$
  2. is invariant under translations
  3. is real and bi-linear
  4. has no unwanted doublers (poles of the propagator other than the physical)
- Loophole: Operators satisfying the Ginsparg-Wilson relation

$$\mathcal{D}\gamma_5 + \gamma_5\mathcal{D} = a\mathcal{D}\gamma_5 R\mathcal{D}$$

obey an exact, lattice modified chiral symmetry

$$\begin{aligned} \gamma_5\mathcal{D} + \mathcal{D}\hat{\gamma}_5 &= 0 \\ \hat{\gamma}_5 &:= \gamma_5\left(1 - aR\mathcal{D}\right) \end{aligned}$$

The overlap operator

The overlap operator

$$\begin{aligned}
\mathcal{D}^{(ov)} &= \frac{\rho}{a}\left\{1 + \frac{A}{\sqrt{A^\dagger A}}\right\} \\
A &:= \mathcal{D}^{(W)} - \frac{\rho}{a}, \quad 0 < \rho < 2r \\
\mathcal{D}^{(W)} &:= \sum_\mu \gamma_\mu \Delta_\mu^s - a\frac{r}{2}\Delta_\mu^b \Delta_\mu^f
\end{aligned}$$

satisfies the Ginsparg-Wilson relation and is used in the following computations.

The euclidean action on the lattice

Finally the fermion action on the euclidean lattice is

$$S_F = a^4 \sum_{x \in \mathbb{Z}_L^4} \overline{\psi}_x \left\{ \mathcal{D}^{(ov)} + yP_+ \Phi_x^\dagger \hat{P}_+ + yP_- \Phi_x \hat{P}_- \right\} \psi_x$$

$$P_\pm := \frac{1}{2}(1 \pm \gamma_5)$$

$$\hat{P}_\pm := \frac{1}{2}(1 \pm \hat{\gamma}_5).$$

The generating functional is given by

$$Z[J] = \int \mathrm{D}\Phi \mathrm{D}\overline{\psi} \mathrm{D}\psi \; e^{-S_\Phi - S_F - J\Phi^0}.$$

Phase diagram

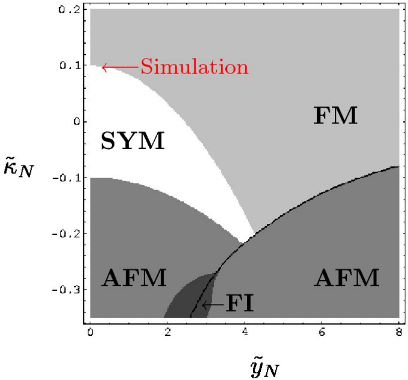


Figure: The phase diagram of the model

Bare parameters

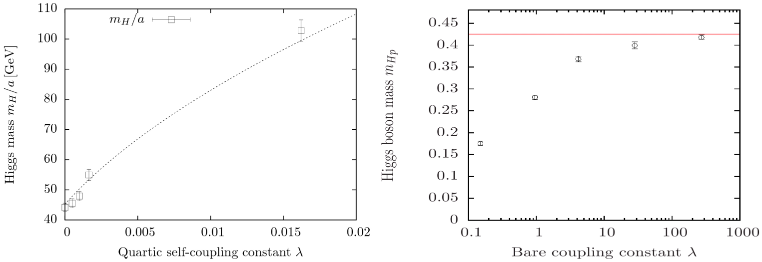
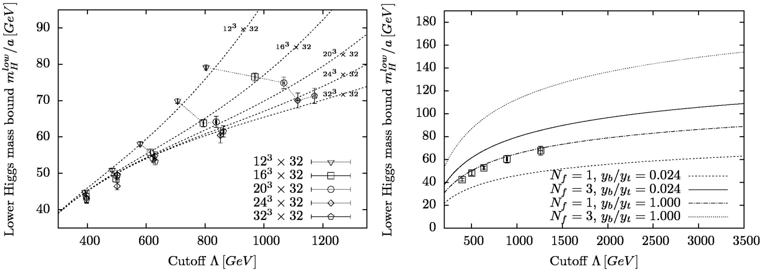


Figure: The figure shows that the lower Higgs boson mass is reached at zero and the the upper bound is reached at infinite bare quartic coupling. For more details see K.Jansen and P.Gerhold arXiv:0902.4135.

The lower mass bound



- The lower Higgs boson mass determined at zero quartic bare coupling and various lattice sizes.
- The dashed curves are obtained by analyzing the effective potential.

The upper Higgs boson mass

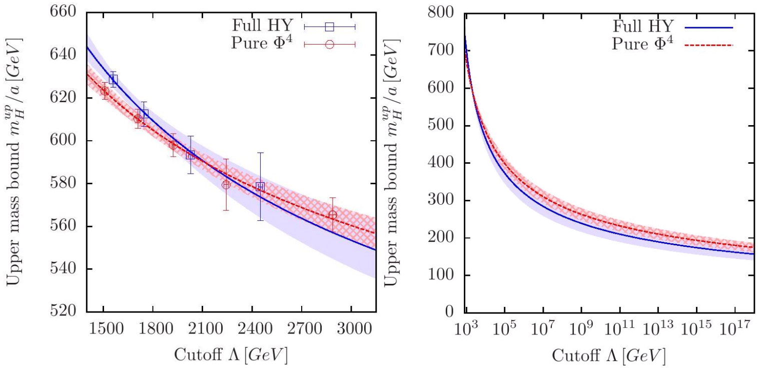


Figure: The upper Higgs boson mass bound obtained at infinite bare quartic coupling. The mass is computed via the pole of the real part of the propagator.

The propagator

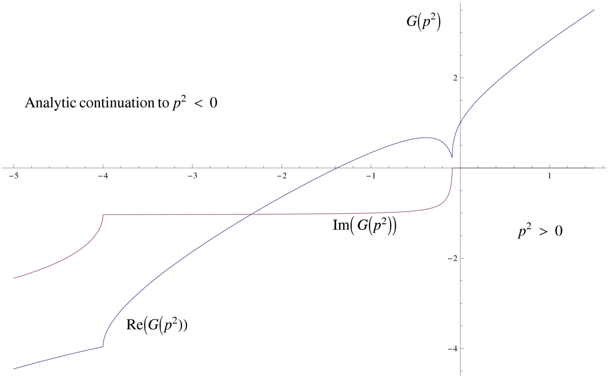


Figure: The euclidean Higgs propagator with its analytic continuation to $p^2 < 0$. There is a pole at $p^2 = m_H^2 + i\Gamma, \Gamma \in \mathbb{R}^+$.

The decay width and the resonance mass

- The Higgs boson is unstable if the spectrum contains light particles.
- The propagator has a complex pole; its real part is the resonance mass, the imaginary part the decay width
- Lattice methods to determine the complex pole:
  1. Analytic continuation of the euclidean propagator.
  2. Finite size analysis of two-particle energies.

Analytic continuation of the euclidean propagator

- A straight forward approach from the theoretical point of view.
- Relies on the (perturbative) functional form of the propagator.
- Requires precise data to obtain reliable results.

κ	λ_0	m_H^C	m_H	Γ	Λ
0.12301	0.0	0.058(2)	0.057(2)	0.0	1163.9 ± 3.6 GeV
0.12313	0.0	0.111(2)	0.108(1)	0.0	393.5 ± 1.3 GeV
0.30039	∞	0.249(6)	0.242(2)	0.035(8)	2373 ± 6.4 GeV
0.30400	∞	0.395(5)	0.408(2)	0.054(7)	1548.1 ± 1.8 GeV

Table: Results obtained from the analytic continuation of the euclidean propagator. All computations have been performed on lattices of extent 32^4.

Conclusions

Comparing the propagator data with the mass obtained by the correlator, we infer:

- The Higgs boson mass can be determined reliably.
- The decay width Γ depends on the fit details.

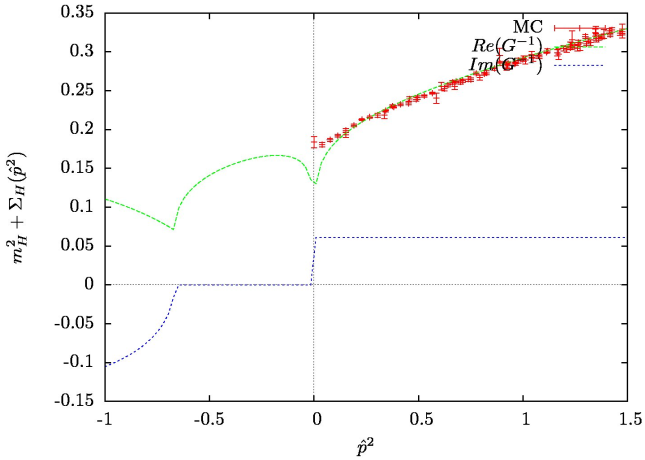


Figure: The euclidean propagator and data from Monte-Carlo simulation at $\kappa = 0.30400, \lambda = \infty$.

The scattering phase

- Close connection between the two-particle energy spectrum in **finite** volume and the elastic scattering amplitude in **infinite** volume (M. Lüscher: Commun. Math. Phys.105, 1986).
- The scattering amplitude in the elastic region is

$$T_J = 16\pi W \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) \underbrace{\frac{1}{2ik}\left(e^{2i\delta_{Jl}-1}\right)}_{\text{partial wave amplitudes}}$$

W is the two particle energy in **finite** volume.

k is a solution of:

$$n\pi - \delta_{Jl}(k) \; = \; \phi(q), \quad q = \frac{kL}{2\pi}$$
$$\tan\phi(q) \; = \; -\frac{\pi^{\frac{3}{2}} q}{\mathcal{Z}_{00}(1; q^2)}.$$

And $\mathcal{Z}_{lm}$ denotes the generalized Zeta function

$$\mathcal{Z}_{lm}(s; q^2) = \sum_{\vec{n} \in \mathbb{Z}^3} \mathcal{Y}_{lm}(\vec{n}) \left(\vec{n}^2 - q^2\right)^{-2}.$$

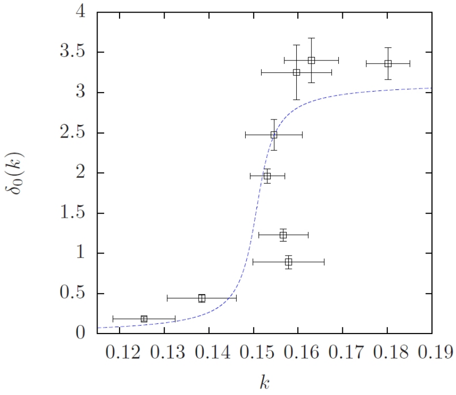


Figure: The figure has been taken from P.Gerhold's PhD thesis.

Conclusion and outlook

- The phase diagram has been computed.
- Non-perturbative lower and upper Higgs boson mass bounds are established.
- The unstable nature of the Higgs boson are discussed from two perspectives
  1. perturbative propagator
  2. non-perturbative method based on finite size effects
- Future computations will involve higher statistics to reduce uncertainties in the scattering phase and the resonance mass.
- The analysis of finite size dependence of two particle energies will be invstigated in a moving frame which allows a finer resolution of relative momenta.