The Chirally rotated Schrödinger Functional and automatic O(a)-improvement

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- Want to study non-perturbative properties of QCD
- Need non-perturbative regularisation
- Lattice regulator: discrete space-time volume, $V = L^3 \times T$, spacing a
  - ultra-violet cut-off $\sim 1/a$
  - observables on the lattice:

    $\langle \mathcal{O} \rangle^{\text{latt}} = \langle \mathcal{O} \rangle^{\text{cont}} + \text{cut-off effects}$
  - remove cut-off $\Leftrightarrow$ continuum limit: $a \to 0$
  - It is very important to reach continuum limit with:

$$\langle \mathcal{O} \rangle^{\text{latt}} = \langle \mathcal{O} \rangle^{\text{cont}} + \mathrm{O}(a^2)$$

better than

$$\langle \mathcal{O} \rangle^{\text{latt}} = \langle \mathcal{O} \rangle^{\text{cont}} + \mathrm{O}(a)$$

- Many different lattice regularisations

- In our case:

  Twisted mass Wilson fermions at maximal twist (R.Frezzotti, G.C.Rossi)

- Some properties:
  - Even-flavour theory
  - Automatic $\mathrm{O}(a)$-improvement:

    All physical quantities computed have only $\mathrm{O}(a^{even})$

    No-need of adding terms to cancel $\mathrm{O}(a)$ effects (improvement)

    with coefficients which would have to be non-perturbatively tuned

$$\langle \mathcal{O} \rangle^{\mathrm{latt}} = \langle \mathcal{O} \rangle^{\mathrm{cont}} + \mathrm{O}(a^2)$$

- From the lattice simulations we obtain bare quantities

$$\mathcal{O}_B^{\mathrm{latt}}(a), \qquad \mathrm{latt} = \text{chosen lattice regularisation} = \mathrm{Wtm}$$

- Many observables need to be renormalised (ren = renormalisation scheme)

$$\mathcal{O}_{\mathrm{R}}^{\mathrm{ren}}(\mu) = \lim_{a\to 0} \mathcal{O}_{\mathrm{R}}^{\mathrm{ren,latt}}(\mu, a) = \lim_{a\to 0} Z_{\mathcal{O}}^{\mathrm{ren,latt}}(\mu, a)\, \mathcal{O}_{\mathrm{B}}^{\mathrm{latt}}(a)$$

- Need a suitable renormalisation scheme to compute $Z_{\mathcal{O}}^{\mathrm{ren,latt}}(\mu, a)$
- Want renormalisation scheme which:
  - is non-perturbative
  - is mass independent
  - preserves automatic $\mathrm{O}(a)$-improvement

$$\mathcal{O}_{\mathrm{R}}^{\mathrm{ren,latt}}(\mu, a) = \mathcal{O}_{\mathrm{R}}^{\mathrm{ren}}(\mu) + O(a^2)$$

(M. Lüscher et al., hep-lat/9207009), (ALPHA)

- Regularisation independent
- Finite-volume: $V = L^3 \times T$
  - Periodic b.c. in the spatial directions

$$\psi(x + L\hat{k}) = \psi(x) \qquad \bar{\psi}(x + L\hat{k}) = \bar{\psi}(x)$$

  - Dirichlet b.c. in the time direction (standard SF b.c.)

$$P_+ \psi(x)|_{x_0=0} = 0 \quad P_- \psi(x)|_{x_0=T} = 0 \quad ; \quad P_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma_0)$$

(S. Sint, hep-lat/9312079, hep-lat/9504005), (M. Lüscher, hep-lat/0603029)

- Can be used at zero mass (non-zero bound of e.v. of Dirac operator $\backsim \frac{1}{T}$)
- Renormalisation scale $\mu = \frac{1}{L}$
- Finite-size techniques: non-perturbative running with $\frac{1}{L}$
- $\mathrm{O}(a)$ effects from the boundaries $\Longrightarrow$ add improvement terms at the boundary

- Start from continuum theory: massless QCD + standard SF b.c.
- Regularise using non-improved massless Wilson fermions
- In a finite volume massless Wilson fermions are automatically $O(a)$-improved provided the original continuum theory is chirally symmetric
- Standard Schrödinger Functional b.c. break chiral symmetry
- A renormalisation scheme with:
  - standard Schrödinger functional boundary conditions
  - non-improved massless Wilson fermions in the bulk
- Incompatible with bulk automatic $O(a)$ improvement:
  need to add improvement terms to the action and operators with coefficients which have to be non-perturbatively tuned
- How to save automatic $O(a)$-improvement using SF schemes with Wilson fermions?
- Chirally rotated Schrödinger Functional (chiSF) (S. Sint, hep-lat/0511034)

- Chiral rotation (in the continuum) of the the quark fields

$$\psi(x) \to e^{i\frac{\alpha}{2}\gamma_5\tau^3}\,\psi(x) \qquad \bar{\psi}(x) \to \bar{\psi}(x)\,e^{i\frac{\alpha}{2}\gamma_5\tau^3} \quad ; \quad \alpha = \pi/2$$

- Brings the standard Schrödinger functional boundary conditions

$$P_+\psi(x)|_{x_0=0} = 0 \qquad P_-\psi(x)|_{x_0=T} = 0 \quad ; \quad P_\pm = \frac{1}{2}\,(\mathbb{1} \pm \gamma_0)$$

- to the chirally rotated SF boundary conditions (S. Sint, hep-lat/0511034)

$$Q_+\psi(x)|_{x_0=0} = 0 \qquad Q_-\psi(x)|_{x_0=T} = 0 \quad ; \quad Q_\pm = \frac{1}{2}\left(\mathbb{1} \pm i\gamma_0\gamma_5\tau^3\right)$$

  - in the continuum both formulations are equivalent
  - chirally rotated SF b.c. are invariant under $\gamma_5\tau^1$–symmetry

- Non-improved massless Wilson fermions in the bulk with chirally rotated SF b.c.
  - compatible with bulk automatic $\mathrm{O}(a)$-improvement

- The regularised theory has less symmetries than the classical continuum theory
- Need to determine operators allowed by the symmetries of the regularised theory
- Irrelevant (d=4) boundary operator (present in any SF formulation on the lattice)
  - boundary $\mathrm{O}(a)$ effects
  - add $d = 4$ boundary counter-term to the action with coefficient d_s

$$\delta S_4 = a(d_s - 1)a^3 \sum_{\vec{x}} \left(\bar{\psi}\gamma_k D_k \psi|_{x_0=0} + \bar{\psi}\gamma_k D_k \psi|_{x_0=T} \right)$$

  - tune d_s to cancel the $\mathrm{O}(a)$ boundary effects
  - perturbative tuning

- Relevant (d=3) boundary operator
  - $\gamma_5\tau^1$-odd: breaking of flavour and parity symmetries
  - incorrect continuum limit
  - induces bulk $O(a)$ effects
  - add $d = 3$ finite boundary counter-term with coefficient z_f

$$\delta S_3 = (z_f - 1)a^3 \sum_{\vec{x}} \left(\bar{\psi}\psi|_{x_0=0} + \bar{\psi}\psi|_{x_0=T} \right)$$

  - need to tune z_f to restore the symmetries at finite lattice spacing

    $\Longrightarrow$ universality and bulk automatic $O(a)$-improvement

- require non-perturbative tuning
  - massless scheme: $m_q = 0 \Leftarrow m_0 \rightarrow m_c \longleftrightarrow \kappa = \dfrac{1}{8 + 2m_0} \rightarrow \kappa_c$
  - universality of the c.l. and bulk improvement: $z_f \rightarrow z_f^*$

Formulation	*standard SF*	*chirally rotated SF*
Projectors	$P_\pm = \frac{1}{2}\left(\mathbb{1} \pm \gamma_0\right)$	$Q_\pm = \frac{1}{2}\left(\mathbb{1} \pm i\gamma_0\gamma_5\tau^3\right)$
Number of Flavours	any	even
Tuning of κ ($m_q = 0$)	yes	yes
In case of using regularisation with Wilson-type fermions		
Boundary d=4 counter-terms	yes	yes
Boundary d=3 counter-terms	no	z_f
Improvement counter-terms to action	yes (c_{SW})	no
Improvement counter-terms to operators	yes	no

- In our case we have
  - non-improved massless Wilson fermions in the bulk
  - chirally rotated SF boundary conditions
- thus we compute:

$$\mathcal{O}_{\mathrm{R}}^{\mathrm{chiSF,Wil}}(\mu, a)$$

- Only need to tune non-perturbatively 2 parameters: κ and z_{f}
- This is enough to guarantee:
  - automatic O(a)-improvement in the bulk:

$$\mathcal{O}_{\mathrm{R}}^{\mathrm{chiSF,Wil}}(\mu, a) = \mathcal{O}_{\mathrm{R}}^{\mathrm{chiSF}}(\mu) + O(a^2)$$

  - universality in the continuum limit:

$$\mathcal{O}_{\mathrm{R}}^{\mathrm{chiSF}}(\mu) = \mathcal{O}_{\mathrm{R}}^{\mathrm{SF}}(\mu)$$

- Quenched approximation of QCD: neglect dynamical quark degrees of freedom
- Keep constant physics: set $T = L$ and fix the renormalisation scale $\mu = 1/L$
- Choose tuning conditions (explained in backup slides)
- Perform tuning for several lattices L/a at fixed renormalisation scale $1/L$
- We have done it for three physical situations:

Results at $\bar{g}^2(L)$ such that $L = 1.436 r_0$ ($E \sim 300$ MeV)

L/a	$\beta = 6/g_0^2$	$z_f^*(g_0, L/a)$	$\kappa_c(g_0, L/a)$
8	6.0219	1.8090(32)	0.153530(24)
10	6.1628	1.7920(30)	0.152134(17)
12	6.2885	1.7664(51)	0.150815(22)
16	6.4956	1.7212(83)	0.148945(25)

Results at $\bar{g}^2(L) = 2.4484$ ($E \sim 1$ GeV)

L/a	$\beta = 6/g_0^2$	$z_f^*(g_0, L/a)$	$\kappa_c(g_0, L/a)$
8	7.0197	1.5467(15)	0.144501(13)
12	7.3551	1.5126(23)	0.143113(12)
16	7.6101	1.4942(37)	0.142112(13)

Results at $\bar{g}^2(L) = 0.9944$ ($E \sim 30$ GeV)

L/a	$\beta = 6/g_0^2$	$z_f^*(g_0, L/a)$	$\kappa_c(g_0, L/a)$
8	10.3000	1.29730(67)	0.1354609(54)
12	10.6086	1.2954(11)	0.1351758(56)
16	10.8910	1.2858(15)	0.1348440(61)

- The scheme is now defined
  - for three physical situations ($\mu = L^{-1} = \text{fix} \Leftrightarrow \bar{g}^2 = \text{fix}$)
  - at several values of the cut-off (several L/a) for each physical situation

- Can compute observables at each physical point for each lattice spacing

$$\mathcal{O}_{\rm R}^{\rm chiSF,Wil}(\mu, a)$$

- Perform the continuum limit at constant physics

$$\mathcal{O}_{\rm R}^{\rm chiSF,Wil}(\mu, a)|_{\mu=\text{fix}} \xrightarrow{a=0} \mathcal{O}_{\rm R}^{\rm chiSF}(\mu)$$

- Scaling towards c.l: leading $\mathrm{O}(a^2)$ cut-off effects. AUTOMATIC-IMPROVEMENT

- Same observables computed with standard SF and np-improved Wilson fermions: e.g. strange quark mass m_s, SSF of Z_P, SSF of $Z_{O_{44}}$ (ALPHA)
  - At finite lattice spacing should not agree: different regularisations
  - In the continuum must agree: SF and chiSF equivalent in the c.l. UNIVERSALITY

- Conclusions:
  - Non-perturbative renormalisation scheme:
    chirally rotated Schrödinger functional
  - Regularisation: massless non-improved Wilson fermions
  - No bulk improvement counter-terms to the action and to operators
  - Only need to tune non-perturbatively m_q and z_f
  - We showed that the tuning is possible
- Outlook:
  - Check universality of the continuum limit
  - Check automatic $\mathrm{O}(a)$-improvement of renormalised quantities
  - Extend this study to four-flavour theory
  - Compute renormalisation constants for physical quantities of interest
    (Important ingredient of ETMC research programme)

- Many different lattice regularisations
- In our case:

  Twisted mass Wilson fermions at maximal twist (R.Frezzotti, G.C.Rossi)

- Lattice fermion action:

$$S^{\mathrm{Wtm}}(\bar{\psi}, \psi, U) = a^4 \sum_x \bar{\psi}(x) \left\{ D_{\mathrm{W}} + i\mu\gamma_5\tau^3 \right\} \psi(x)$$

$$D_{\mathrm{W}} = \frac{1}{2} \left\{ \gamma_\mu \left(\nabla_\mu + \nabla^*_\mu \right) - a\nabla^*_\mu \nabla_\mu \right\}$$

- Some properties:
  - Two flavour theory
  - Massless twisted mass Wilson fermions = massless standard Wilson fermions
  - Automatic $\mathrm{O}(a)$-improvement

$$\langle \mathcal{O} \rangle^{\mathrm{latt}} = \langle \mathcal{O} \rangle^{\mathrm{cont}} + \mathrm{O}(a^2)$$

- The action: (S. Sint)

$$S[\bar{\psi}, \psi, U] = a^4 \sum_{x_0=0}^{T} \sum_{\vec{x}} \bar{\psi}(x) \mathcal{D} \psi(x)$$

- Lattice Dirac operator:

$$a\mathcal{D}\psi(x) = \begin{cases} -U_0(x)P_-\psi(x+a\hat{0}) + \left[aK + i\gamma_5\tau^3 P_-\right]\psi(x) & x_0 = 0 \\ aD_W\psi(x) & 0 < x_0 < T \\ \left[aK + i\gamma_5\tau^3 P_+\right]\psi(x) - U_0^\dagger(x-a\hat{0})P_+\psi(x-a\hat{0}) & x_0 = T \end{cases}$$

  - Wilson operator:

$$aD_W\psi(x) = -U_0(x)P_-\psi(x+a\hat{0}) + aK\psi(x) - U_0^\dagger(x-a\hat{0})P_+\psi(x-a\hat{0})$$

  - Spatial part of the Wilson operator:

$$aK\psi(x) = \left[\frac{1}{2\kappa} - 3 + \frac{1}{2}\sum_{k=1}^{3}\left\{a\gamma_k\left(\nabla_k + \nabla_k^*\right) - a^2\nabla_k^*\nabla_k\right\}\right]\psi(x);\ 2\kappa = (4+am)^{-1}$$

- Keep constant physics: set $T = L$ and fix the renormalisation scale $\mu = 1/L$

  case 1: $L = 1.436 r_0$ matching scale with HS $(E \sim 300\ \mathrm{MeV})$

  case 2: $\bar{g}^2 = 0.9944$ matching scale with PT $(E \sim 30\ \mathrm{GeV})$

  case 3: $\bar{g}^2 = 2.4484$ intermediate scale $(E \sim 1\ \mathrm{GeV})$

- Set: $d_s = d_s^{(0)} = 1/2$

- Tuning of κ: $m_{PCAC} \equiv \dfrac{\partial_0^{\mathrm{latt}} g_{A\_}^{11}(T/2a)}{2 g_{P\_}^{11}(T/2a)} = 0$

- Tuning of z_f: $g_{A\_}^{11}(T/2a) = 0$

- Reason for choosing these tuning conditions:

$$\gamma_5 \tau^1 - \text{odd boundary to bulk c.f:} \quad g_{A_\pm}^{11}(x_0) = -\langle A_0^1(x) \mathcal{P}_\pm^1 \rangle \Longrightarrow \text{vanishing c.l.}$$

$$\gamma_5 \tau^1 - \text{even boundary to bulk c.f:} \quad g_{P_\pm}^{11}(x_0) = -\langle P^1(x) \mathcal{P}_\pm^1 \rangle \Longrightarrow \text{non-vanishing c.l.}$$

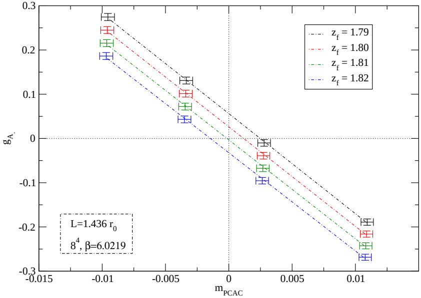
$$g_A^{ab}(x_0)_\pm = -\langle A_0^a(x)\mathcal{P}_\pm^b\rangle \qquad g_P^{ab}(x_0)_\pm = -\langle P^a(x)\mathcal{P}_\pm^b\rangle$$

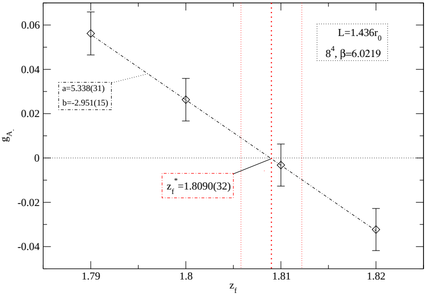
$$\mathcal{P}_\pm^a = a^6 \sum_{\vec{y},\vec{z}} \bar{\zeta}(\vec{y})\gamma_5 \frac{1}{2}\tau^a Q_\pm \zeta(\vec{z})\, e^{i\vec{p}(\vec{y}-\vec{z})}$$

$$Q_\pm = \frac{1}{2}\left(\mathbb{1} \pm i\gamma_0\gamma_5\tau^3\right)$$

$$\zeta(\vec{x}) = U(x_0-a,\vec{x};0)\psi(x)|_{x_0=a} \qquad \bar{\zeta}(\vec{x}) = \bar{\psi}(x)U^\dagger(x_0-a,\vec{x};0)|_{x_0=a}$$

- Tuning procedure at fixed renormalisation scale $L = 1.436 r_0$ and lattice size $L/a = 8$



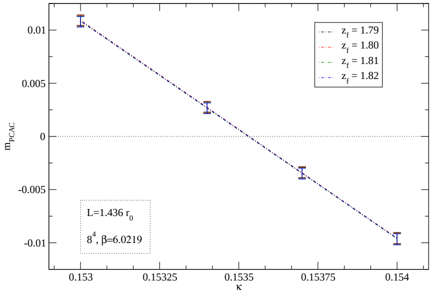


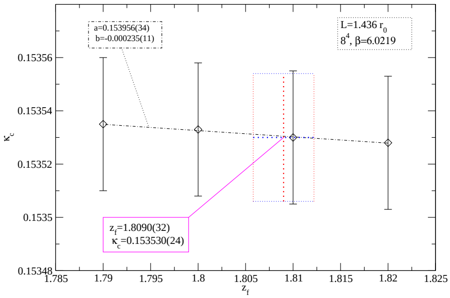
$$\frac{dg_A}{dm_{PCAC}} \sim -22.5$$

$$\mathrm{rerr} \sim 0.5\%$$

$$z_f^* = 1.8090(32) \quad \mathrm{rerr} \sim 0.2\%$$

$$\frac{dg_A}{dz_f} = -2.951(14) \quad \mathrm{rerr} \sim 0.5\%$$





$$\frac{dm_{PCAC}}{d\kappa} \sim -20.5$$

$$\text{rerr} \sim 0.2\%$$

$$\kappa_c = 0.153530(24) \quad \text{rerr} \sim 0.02\%$$

$$\frac{d\kappa_c}{dz_f} = -0.000235(11) \quad \text{rerr} \sim 5\%$$

- Parity

$$\mathcal{P}:\begin{cases}\chi(x) \longrightarrow \gamma_0\,\chi(x_0,-\vec{x})\\ \bar{\chi}(x) \longrightarrow \bar{\chi}(x_0,-\vec{x})\,\gamma_0\end{cases}\qquad \mathcal{P}_F^{1,2}:\begin{cases}\chi(x) \longrightarrow i\,\gamma_0\,\tau^{1,2}\,\chi(x_0,-\vec{x})\\ \bar{\chi}(x) \longrightarrow -i\,\bar{\chi}(x_0,-\vec{x})\,\gamma_0\,\tau^{1,2}\end{cases}$$

- Chiral symmetry and twisted chiral symmetry

$$SU_V(2):\begin{cases}\chi(x)\rightarrow e^{i\frac{\alpha_V}{2}\tau^a}\chi(x)\\ \bar{\chi}(x)\rightarrow \bar{\chi}(x)e^{-i\frac{\alpha_V}{2}\tau^a}\end{cases}\quad SU_V(2)_\omega:\begin{cases}\chi(x)\rightarrow e^{-i\frac{\omega}{2}\gamma_5\tau^3}e^{i\frac{\alpha_V}{2}\tau^a}e^{i\frac{\omega}{2}\gamma_5\tau^3}\chi(x)\\ \bar{\chi}(x)\rightarrow \bar{\chi}(x)e^{i\frac{\omega}{2}\gamma_5\tau^3}e^{-i\frac{\alpha_V}{2}\tau^a}e^{-i\frac{\omega}{2}\gamma_5\tau^3}\end{cases}$$

$$SU_A(2):\begin{cases}\chi(x)\rightarrow e^{i\frac{\alpha_A}{2}\gamma_5\tau^a}\chi(x)\\ \bar{\chi}(x)\rightarrow \bar{\chi}(x)e^{i\frac{\alpha_A}{2}\gamma_5\tau^a}\end{cases}\quad SU_A(2)_\omega:\begin{cases}\chi(x)\rightarrow e^{-i\frac{\omega}{2}\gamma_5\tau^3}e^{i\frac{\alpha_A}{2}\gamma_5\tau^a}e^{i\frac{\omega}{2}\gamma_5\tau^3}\chi(x)\\ \bar{\chi}(x)\rightarrow \bar{\chi}(x)e^{i\frac{\omega}{2}\gamma_5\tau^3}e^{i\frac{\alpha_A}{2}\gamma_5\tau^a}e^{-i\frac{\omega}{2}\gamma_5\tau^3}\end{cases}$$

- Charged subgroup of twisted chiral symmetry at $\omega = \frac{\pi}{2}$

$$\left[U_V(1)_{\frac{\pi}{2}}\right]_{1,2}:\begin{cases}\chi(x)\rightarrow e^{\pm i\frac{\alpha_V}{2}\gamma_5\tau^{2,1}}\chi(x)\\ \bar{\chi}(x)\rightarrow \bar{\chi}(x)e^{\pm i\frac{\alpha_V}{2}\gamma_5\tau^{2,1}}\end{cases}\qquad \left[U_A(1)_{\frac{\pi}{2}}\right]_{1,2}:\begin{cases}\chi(x)\rightarrow e^{\pm i\frac{\alpha_A}{2}\tau^{2,1}}\chi(x)\\ \bar{\chi}(x)\rightarrow \bar{\chi}(x)e^{\mp i\frac{\alpha_A}{2}\tau^{2,1}}\end{cases}$$