5-point integrals and Mellin-Barnes method

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  - Real soft IR divergence
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  - QCD massless 5-point function
  - QCD massive 5-point function
  - Binomial sums and PSQL
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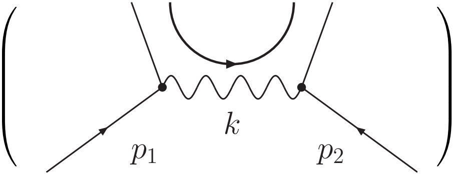
IR divergence in Feynman integrals

$\int d^4k$ 

Virtual corrections IR singularity:

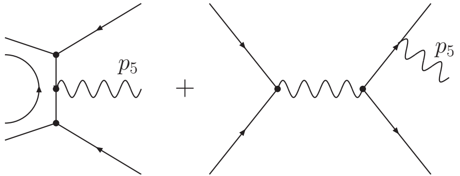
$$\int \frac{d^4k}{\left((k-p_1)^2 - m_1^2\right) k^2 \left((k+p_2)^2 - m_2^2\right)} =$$

$$= \int \frac{d^4k}{\left(k^2 - 2kp_1\right) k^2 \left(k^2 + 2kp_2\right)} \sim \int \frac{dk}{k} \longrightarrow \text{IR divergent}$$

- Contribute to $1/\epsilon^n$ terms in Laurent expansion.
- Naturally separated from the finite parts.

IR singularities in Feynman integrals

$$\int \frac{d^3p_5}{2E_5} \left| \quad \cdots \quad \right|^2$$



Real soft massless emission IR singularity:

$$\int \frac{d^3p_5}{2E_5}\frac{A}{E_5}\frac{B(E_5)}{E_5} \longrightarrow \int_0^\omega \frac{dE_5}{E_5} \longrightarrow \text{IR divergent}$$

$$\int_0^\omega \frac{dE_5}{E_5^{4-d}} \overbrace{\left(\frac{a}{\epsilon E_5} + \frac{b\ln(E_5)}{E_5} + \frac{c}{E_5}\right)} = -\frac{2a+b}{4\epsilon^2} - \frac{c-2a\ln(\omega)}{2\epsilon} + O(1)$$

- Parts $\sim 1/E_5$, $\ln(E_5)/E_5$ — contribute to $1/\epsilon^n$ terms **after** phase-space integration.
- Hidden in ϵ-finite parts of matrix element.

Intermediate summary

Two kinds of IR singularities:

"Virtual"

- Localized in $1/\epsilon^n$ terms.
- Can be calculated with any method.

"Real"

- **Not** localized
  (may be found in constant term!).
- May change order of virtual singularities.
- Separation from the finite part is **not** obvious.

Solution: Mellin-Barnes method

Example of mixed IR singularity

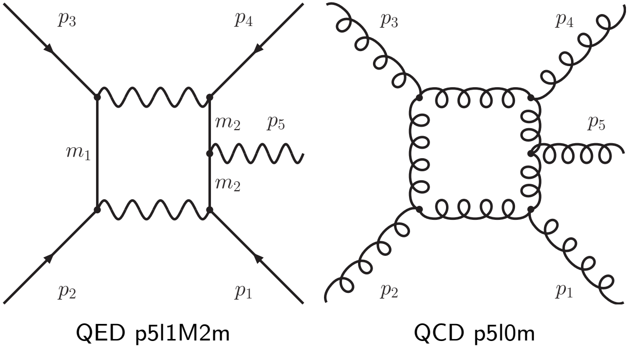


Figure: Mixed virtual/real IR singular 5-point functions

5 independent kinematic invariants:

$$p_1 p_2 = s'_{12}/2 \qquad p_1 p_5 = v_{15}/2 \sim E_5$$

$$p_3 p_4 = s'_{34}/2 \qquad p_4 p_5 = v_{45}/2 \sim E_5$$

$$(p_2 + p_3)^2 = s_{23}$$

Definitions

Massive one-loop n-point scalar Feynman integral:

$$I = \frac{e^{\epsilon\gamma_E}}{i\pi^{d/2}} \int \frac{d^d k}{(q_1^2 - m_1^2)^{\nu_1}(q_2^2 - m_2^2)^{\nu_2} \dots (q_n^2 - m_n^2)^{\nu_n}}$$

Feynman parameters representation ($\nu = \sum_{i=1}^{N} \nu_i$):

$$I = \frac{e^{\epsilon\gamma_E}(-1)^{\nu}\Gamma(\nu - d/2)}{\prod_{i=1}^{N} \Gamma(\nu_i)} \left(\prod_{j=1}^{N} \int_0^1 dx_j x_j^{\nu_j - 1} \right) \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{U^{\nu-d}}{F^{\nu-d/2}}$$

In one-loop $U = \sum_{i=1}^{N} x_i = 1$,
and F could be made bilinear in x_i.

Mellin-Barnes formula

Muon pair production p5l1M2m F-form:

$$F_{\text{p5l1M2m}} = m_2^2(x_1+x_5)^2 + m_1^2 x_3^2 - \\ - s'_{12}x_1x_3 - s'_{34}x_3x_5 - v_{45}x_1x_4 - v_{15}x_2x_5 - s_{23}x_2x_4$$

Massless QCD p5l0m F-form:

$$F_{\text{p5l0m}} = -s_{12}x_1x_3 - s_{34}x_3x_5 - v_{45}x_1x_4 - v_{15}x_2x_5 - s_{23}x_2x_4$$

Mellin-Barnes formula

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Mellin-Barnes formula and generalized Beta-function

$$\frac{1}{(X+Y)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty+R}^{+i\infty+R} dz\, \Gamma(\lambda+z)\Gamma(-z)\frac{Y^z}{X^{\lambda+z}}$$

$$\prod_{j=1}^{N}\int_0^1 dx_j x_j^{\nu_j-1}\delta\left(1-x_1-\cdots-x_N\right) = \frac{\Gamma(\nu_1)\Gamma(\nu_2)\cdots\Gamma(\nu_N)}{\Gamma(\nu_1+\nu_2+\cdots+\nu_N)}$$

Mellin-Barnes formula

Muon pair production p5l1M2m F-form: (**7-d integral** $\rightarrow$ **6-d**)

$$F_{\text{p5l1M2m}} = m_2^2(x_1+x_5)^2 + m_1^2 x_3^2 - \\ - s'_{12}x_1x_3 - s'_{34}x_3x_5 - v_{45}x_1x_4 - v_{15}x_2x_5 - s_{23}x_2x_4$$

Massless QCD p5l0m F-form: (**4-d integral**)

$$F_{\text{p5l0m}} = -s_{12}x_1x_3 - s_{34}x_3x_5 - v_{45}x_1x_4 - v_{15}x_2x_5 - s_{23}x_2x_4$$

Mellin-Barnes formula and generalized Beta-function

$$\frac{1}{(X+Y)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty+R}^{+i\infty+R} dz\, \Gamma(\lambda+z)\Gamma(-z)\frac{Y^z}{X^{\lambda+z}}$$

$$\prod_{j=1}^{N}\int_0^1 dx_j x_j^{\nu_j-1}\delta\left(1-x_1-\cdots-x_N\right) = \frac{\Gamma(\nu_1)\Gamma(\nu_2)\cdots\Gamma(\nu_N)}{\Gamma(\nu_1+\nu_2+\cdots+\nu_N)}$$

MB representation for p5l1M2m

- MB integrals with *AMBRE* (J. Gluza, K. Kajda, T. Riemann).
- Continuation algorithm by J. B. Tausk (hep-ph/9909506).
- Algorithm implemented in *MB.m* [1] (M. Czakon).

IR kinematics $s'_{12} \approx s'_{34}$ + Barnes 1st lemma $\to -1$ integration.

$$I = ({m_1}^2)^{z_3}({m_2}^2)^{z_1}(-s_{23})^{-\epsilon-z_1-z_3-z_4-z_5-z_6}(-s'_{12})^{z_5}(-v_{15})^{z_6}(-v_{45})^{z_4}$$
$$\Gamma(-z_1)\Gamma(-z_3)\Gamma(-z_4)\Gamma(-z_5)\Gamma(-z_6)\Gamma(z_4+1)\Gamma(z_6+1)\Gamma(2z_3+z_5+1)$$
$$\Gamma(-\epsilon-z_1-z_3-z_5-z_6-2)\Gamma(-\epsilon-z_1-z_3-z_4-z_5-2)\Gamma(2z_1+z_4+z_5+z_6+2)$$
$$\Gamma(\epsilon+z_1+z_3+z_4+z_5+z_6+3)/\left({s_{23}}^3\Gamma(-2\epsilon-1)\Gamma(z_4+z_6+2)\right)$$

Integration contours parallel to imaginary axis:

$\epsilon = -3/4$	$\Re z_1 = -1/2$	$\Re z_3 = -3/16$
$\Re z_4 = -3/32$	$\Re z_5 = -7/16$	$\Re z_6 = -31/64$

[1] with modifications

Continuation for p5l1M2m

Default parameters result (in MB.m notation):

$$\Big\{\text{MBint}\Big[({m_1}^2)^{z_3}({m_2}^2)^{z_3-z_6}(-s_{23})^{-z_6-2}s_{23}(-s'_{12})^{-2z_3-1}(-v_{15})^{z_6}(-v_{45})^{z_6-1}\Gamma(-z_3)$$

$$\Gamma(2z_3+1)\Gamma(1-z_6)\Gamma(-z_6)\Gamma(z_6)\Gamma(z_6+1)^2\Gamma(z_6-z_3)/(\Gamma(2z_6+1)),\Big\{\{\epsilon\to 0\},\Big\{z_3\to -\tfrac{3}{16},z_6\to -\tfrac{11}{32}\Big\}\Big\}\Big],$$

$$\text{MBint}\Big[({m_1}^2)^{z_6}(-s_{23})^{-z_6-2}s_{23}(-s'_{12})^{-2z_6-1}(-v_{15})^{z_6}(-v_{45})^{z_6-1}\Gamma(1-z_6)\Gamma(-z_6)^2\Gamma(z_6)$$

$$\Gamma(z_6+1)^2,\Big\{\{\epsilon\to 0\},\Big\{z_6\to -\tfrac{11}{32}\Big\}\Big\}\Big],\text{MBint}\Big[\Big(({m_1}^2)^{z_3}({m_2}^2)^{z_3}(-s'_{12})^{-2z_3-1}\Gamma(-z_3)^2\Gamma(2z_3+1)$$

$$(-\epsilon\ln(-s_{23})v_{15}+2\epsilon\ln(-v_{15})v_{15}+\epsilon\gamma v_{15}+v_{15}-\epsilon\gamma v_{45}+v_{45}+\epsilon(v_{45}-v_{15})\ln({m_2}^2)+\epsilon v_{45}\ln(-s_{23})-$$

$$-2\epsilon v_{45}\ln(-v_{15})+\epsilon(v_{15}-v_{45})\psi(-z_3))\Big)/(2\epsilon s_{23}v_{15}v_{45}),\Big\{\{\epsilon\to 0\},\Big\{z_3\to -\tfrac{3}{16}\Big\}\Big\}\Big]\Big\}$$

Continuation for p5l1M2m

Default parameters result (in MB.m notation):

$$\Big\{\text{MBint}\Big[(m_1{}^2)^{z_3}(m_2{}^2)^{z_3-z_6}(-s_{23})^{-z_6-2}s_{23}(-s'_{12})^{-2z_3-1}(-v_{15})^{z_6}(-v_{45})^{z_6-1}\Gamma(-z_3)\Gamma(2z_3+1)\Gamma(1-z_6)\Gamma(-z_6)\Gamma(z_6)\Gamma(z_6+1)^2\Gamma(z_6-z_3)/(\Gamma(2z_6+1)),\Big\{\{\epsilon\to 0\},\Big\{z_3\to -\tfrac{3}{16},z_6\to -\tfrac{11}{32}\Big\}\Big\}\Big],$$

$$\text{MBint}\Big[(m_1{}^2)^{z_6}(-s_{23})^{-z_6-2}s_{23}(-s'_{12})^{-2z_6-1}(-v_{15})^{z_6}(-v_{45})^{z_6-1}\Gamma(1-z_6)\Gamma(-z_6)^2\Gamma(z_6)\Gamma(z_6+1)^2,\Big\{\{\epsilon\to 0\},\Big\{z_6\to -\tfrac{11}{32}\Big\}\Big\}\Big],\text{MBint}\Big[\Big((m_1{}^2)^{z_3}(m_2{}^2)^{z_3}(-s'_{12})^{-2z_3-1}\Gamma(-z_3)^2\Gamma(2z_3+1)$$

$$(-\epsilon\ln(-s_{23})v_{15}+2\epsilon\ln(-v_{15})v_{15}+\epsilon\gamma v_{15}+v_{15}-\epsilon\gamma v_{45}+v_{45}+\epsilon(v_{45}-v_{15})\ln(m_2{}^2)+\epsilon v_{45}\ln(-s_{23})-$$

$$-2\epsilon v_{45}\ln(-v_{15})+\epsilon(v_{15}-v_{45})\psi(-z_3))\Big)/(2\epsilon s_{23}v_{15}v_{45}),\Big\{\{\epsilon\to 0\},\Big\{z_3\to -\tfrac{3}{16}\Big\}\Big\}\Big]\Big\}$$

"Optimized continuation" result:

$$I_{\text{p5l1M2m(IR)}} = -(s_{23}s'_{12})^{-1}(m_2^2)^{\epsilon}\left((-v_{45})^{-2\epsilon-1}+(-v_{15})^{-2\epsilon-1}\right)\times$$
$$\times\frac{1}{2}\int dz_3(m_1^2)^{z_3}(m_2^2)^{z_3}(-s'_{12})^{-2z_3}\times$$
$$\times\Gamma(-z_3)^2\Gamma(2z_3+1)\big(-1/\epsilon+\gamma+\psi(-z_3)\big)$$

where $\Re z_3 = -3/16$

Both representations are equivalent (checked numerically).

QED p5l1M2m IR part

$$I_{\text{p5l1M2m(IR)}} = -(s_{23}s'_{12})^{-1}(m_2^2)^{\epsilon} \times$$
$$\times \left((-v_{45})^{-2\epsilon-1} + (-v_{15})^{-2\epsilon-1}\right)\left(\frac{J_{-1}}{\epsilon} + J_0\right)$$

Sums: hep-th/0303162v4 (A. I. Davydychev, M. Yu. Kalmykov).

$$J_{-1} = \sum_{n=0}^{\infty} u^n \binom{2n}{n}\Big(S_1(2n) - S_1(n) + \ln(u)/2\Big)$$
$$J_0 = \sum_{n=0}^{\infty} u^n \binom{2n}{n}\Big(S_2(2n) - S_1(2n)^2 + S_1(2n)S_1(n) -$$
$$- S_1(2n)\ln(u) + S_1(n)\ln(u)/2 - \left(\ln^2(u)/4 + \zeta_2\right)\Big)$$

where $u = m_1^2 m_2^2 / {s'_{12}}^2$

QED p5l1M2m IR part

$$I_{\text{p5l1M2m(IR)}} = -(s_{23}s'_{12})^{-1}(m_2^2)^{\epsilon} \times$$
$$\times \left((-v_{45})^{-2\epsilon-1} + (-v_{15})^{-2\epsilon-1}\right)\left(\frac{J_{-1}}{\epsilon} + J_0\right)$$

Analytical result for IR-part of p5l1M2m:

$$J_{-1} = \frac{1}{2}\frac{(1+\chi)\ln(\chi)}{1-\chi}, \qquad \chi = \frac{1-\sqrt{1-4{m_1}^2{m_2}^2/{s'_{12}}^2}}{1+\sqrt{1-4{m_1}^2{m_2}^2/{s'_{12}}^2}},$$

$$J_0 = \frac{1}{4(1-\chi)}\Big(2(1-\chi)\mathrm{Li}_2(\chi^2) + 8\chi\mathrm{Li}_2(\chi) - 4(1-\chi)\mathrm{Li}_2(-\chi) +$$
$$+ 4(1+\chi)\ln(1-\chi)\ln(\chi) - (1+\chi)\ln^2(\chi) - 4(1+\chi)\zeta_2\Big)$$

Hypergeometric representation

Integral before ϵ-expansion:

$$I_{\text{p5l1M2m(IR)}} = -(s_{23}s'_{12})^{-1}(m_2^2)^{\epsilon}\left((-v_{45})^{-2\epsilon-1} + (-v_{15})^{-2\epsilon-1}\right) \times$$
$$\times \int dz_3 (m_1^2)^{z_3}(m_2^2)^{z_3}(-s'_{12})^{-2z_3} \times$$
$$\times \frac{\Gamma(-2\epsilon)\Gamma(1+2\epsilon)\Gamma(-\epsilon - z_3)\Gamma(-z_3)\Gamma(1+2z_3)}{\Gamma(1-2\epsilon)}$$

Hypergeometric representation (only IR part), $u = m_1^2 m_2^2 / {s'_{12}}^2$

$$I_{\text{p5l1M2m(IR)}} = -(s_{23}s'_{12})^{-1}(m_2^2)^{\epsilon}\left((-v_{45})^{-2\epsilon-1} + (-v_{15})^{-2\epsilon-1}\right) \times$$
$$\times \frac{1}{2}\Gamma(2\epsilon)\big[-2^{1-2\epsilon}u^{-\epsilon}(4u-1)^{\epsilon-\frac{1}{2}}\sqrt{\pi}\Gamma\left(1/2-\epsilon\right) +$$
$$+ u^{-1}\Gamma(1-\epsilon)\,{}_2F_1\left(1, 1-\epsilon; 3/2; 1/(4u)\right)\big]$$

Automatic expansion with *HypExp2* (T. Huber, D. Maitre).

Massless QCD 5l0m

MB representation is 3-dimensional

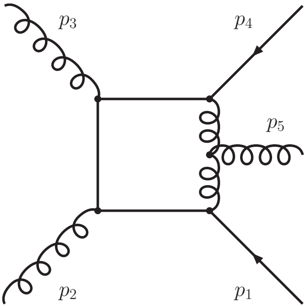
$$I_{\mathsf{p5l0m}} = (-s_{12})^{z_3}(-s_{23})^{-z_2-z_3-z_4}(-v_{15})^{z_4}(-v_{45})^{z_2}\Gamma(-z_2)\Gamma(z_2+1)$$
$$\Gamma(-z_3)\Gamma(z_3+1)\Gamma(-z_4)\Gamma(z_4+1)\Gamma(z_2+z_3+z_4+2)\Gamma(-z_2-z_3-\epsilon-2)$$
$$\Gamma(-z_3-z_4-\epsilon-2)\Gamma(z_2+z_3+z_4+\epsilon+3)/({s_{23}}^3\Gamma(z_2+z_4+2)\Gamma(-2\epsilon-1))$$

$$\epsilon=-1, \quad \Re z_2 = -1/2, \quad \Re z_3 = -13/16, \quad \Re z_4 = -7/16$$

The result for IR part:

$$I_{\mathsf{p5l0m(IR)}} = -\frac{1}{s_{12}}(-s_{23})^{-1-\epsilon}\Bigg[(-s_{23})^{1+2\epsilon}(-v_{15})^{-1-\epsilon}(-v_{45})^{-1-\epsilon}\left(\frac{2}{\epsilon^2}+\zeta_2+\epsilon\frac{14}{3}\zeta_3\right)$$
$$+\left(\frac{1}{\epsilon^2}+\frac{5\zeta_2}{2}\right)\left((-v_{45})^{-1-\epsilon}(-v_{15})^{\epsilon}+(-v_{15})^{-1-\epsilon}(-v_{45})^{\epsilon}\right)+\frac{1}{\epsilon^2}(-s_{12})^{-1-2\epsilon}(-v_{15})^{\epsilon}(-v_{45})^{\epsilon}$$
$$+\frac{\left(v_{15}\ln^2\left(\frac{v_{15}}{s_{23}}\right)+2(s_{23}-v_{15})\ln\left(1-\frac{v_{15}}{s_{23}}\right)\ln\left(\frac{v_{15}}{s_{23}}\right)+4\zeta_2 v_{15}+2(s_{23}-v_{15})\mathsf{Li}_2\left(\frac{v_{15}}{s_{23}}\right)\right)}{v_{15}v_{45}}$$
$$+\frac{\left(v_{45}\ln^2\left(\frac{v_{45}}{s_{23}}\right)+2(s_{23}-v_{45})\ln\left(1-\frac{v_{45}}{s_{23}}\right)\ln\left(\frac{v_{45}}{s_{23}}\right)+4\zeta_2 v_{45}+2(s_{23}-v_{45})\mathsf{Li}_2\left(\frac{v_{45}}{s_{23}}\right)\right)}{v_{15}v_{45}}\Bigg]$$

QCD p5l3m



$$F = m^2(x_2 + x_3 + x_4)^2 - s'_{12}x_1x_3 - s'_{34}x_3x_5 - \\ - v_{45}x_1x_4 - v_{15}x_2x_5 - s_{23}x_2x_4$$

$$I_{\text{p5l3m}} = (m^2)^{z_1} (-s_{23})^{-3-z_1-z_4-z_5-z_6}(-s'_{12})^{z_5}(-v_{15})^{z_6}(-v_{45})^{z_4}\Gamma(-z_1) \\ \Gamma(-z_4)\Gamma(z_4+1)\Gamma(-z_5)\Gamma(z_5+1)\Gamma(-z_6)\Gamma(z_6+1)\Gamma(-z_1-z_5-z_6-\epsilon-2) \\ \Gamma(-z_4-z_5-z_6-2\epsilon-3)\Gamma(-z_1-z_4-z_5-\epsilon-2)\Gamma(z_1+z_4+z_5+z_6+\epsilon+3) \\ \Gamma(z_4+z_5+z_6+2)/(\Gamma(z_4+z_6+2)\Gamma(-2\epsilon-1)\Gamma(-2z_1-z_4-z_5-z_6-2\epsilon-3))$$

QCD p5l3m

$$I = \mathsf{MBint}\Big(-2\,(m^2)^{z_1}\,(-s_{23})^{-z_1}(-v_{15})^{z_6}(-v_{45})^{-2-2\epsilon-z_6}\Gamma(-z_1)$$
$$\Gamma_B(-z_6-1)\Gamma_A(-z_1-z_6-1)\Gamma(-z_6)\Gamma(z_6+1)\Gamma(z_6+2)\Gamma(-z_1+z_6+1)$$
$$\Gamma(z_1)/(s'_{12}\Gamma(-2z_1)),\{\{\epsilon\to 0\},\{z_1\to -87/128, z_6\to -5/64\}\}\Big)$$

"Bad" power of v_{45}:

$$(-v_{45})^{-2-2\epsilon-z_6} \longrightarrow (-v_{45})^{-1-2\epsilon-(-5/64)}$$

Shift contour and take residues in $z_6 = -1 - z_1$ and $z_6 = -1$.

$$I = \mathsf{Res}_A + \mathsf{Res}_B + I_{\mathsf{shifted}}$$

I_{shifted} — IR safe: $(-v_{45})^{-1-2\epsilon-(-1-5/64)}$
Res_1 and Res_2 — one dimension less.

IR part of QCD p5l3m

Analytical result for IR part of QCD p5l3m function *(preliminary)*.

$$
\begin{aligned}
I_{\text{p5l3m(IR)}} = & -\frac{4(-s_{23})^{2\epsilon}\sin^{-1}\left(\frac{\sqrt{s_{23}}}{2m}\right)^2(-v_{15})^{-2\epsilon-1}}{s'_{12}v_{45}} + \frac{\left(m^2\right)^{\epsilon}(-s_{23})^{\epsilon}\ln\left(\frac{v_{45}}{v_{15}}\right)(-v_{15})^{-2\epsilon-1}}{s'_{12}v_{45}\epsilon} \\
& + \frac{7\left(m^2\right)^{\epsilon}\pi^2(-s_{23})^{\epsilon}(-v_{15})^{-2\epsilon-1}}{12s'_{12}v_{45}} - \frac{\left(m^2\right)^{\epsilon}(-s_{23})^{\epsilon}(-v_{15})^{-2\epsilon-1}}{s'_{12}v_{45}\epsilon^2} \\
& - \frac{\epsilon\ln^3\left(-\frac{s_{23}}{m^2}\right)}{3s'_{12}v_{15}v_{45}} - \frac{\epsilon\ln^3\left(\frac{v_{15}}{v_{45}}\right)}{s'_{12}v_{15}v_{45}} - \frac{\epsilon\ln\left(-\frac{s_{23}}{m^2}\right)\ln^2\left(\frac{v_{15}}{v_{45}}\right)}{s'_{12}v_{15}v_{45}} - \frac{\epsilon\ln^2\left(-\frac{s_{23}}{m^2}\right)\ln\left(\frac{v_{15}}{v_{45}}\right)}{s'_{12}v_{15}v_{45}} - \frac{3\pi^2\epsilon\ln\left(\frac{v_{15}}{v_{45}}\right)}{s'_{12}v_{15}v_{45}} \\
& - \frac{(-s_{23})^{2\epsilon}(-v_{45})^{-2\epsilon-1}\epsilon\ln\left(1-\frac{s_{23}v_{15}}{m^2v_{45}}\right)\ln^2\left(-\frac{m^2v_{45}}{s_{23}v_{15}}\right)}{s'_{12}v_{15}} - \frac{\pi^2\epsilon\ln\left(-\frac{s_{23}}{m^2}\right)}{s'_{12}v_{15}v_{45}} \\
& - \frac{\pi^2(-s_{23})^{2\epsilon}(-v_{45})^{-2\epsilon-1}\epsilon\ln\left(1-\frac{s_{23}v_{15}}{m^2v_{45}}\right)}{s'_{12}v_{15}} - \frac{8(-s_{23})^{2\epsilon}(-v_{45})^{-2\epsilon-1}\epsilon\sin^{-1}\left(\frac{\sqrt{s_{23}}}{2m}\right)^2\ln\left(\frac{v_{15}}{v_{45}}\right)}{s'_{12}v_{15}} \\
& + \frac{2(-s_{23})^{2\epsilon}(-v_{45})^{-2\epsilon-1}\epsilon\ln\left(-\frac{m^2v_{45}}{s_{23}v_{15}}\right)\text{Li}_2\left(\frac{s_{23}v_{15}}{m^2v_{45}}\right)}{s'_{12}v_{15}} + \frac{2(-s_{23})^{2\epsilon}(-v_{45})^{-2\epsilon-1}\epsilon\text{Li}_3\left(\frac{s_{23}v_{15}}{m^2v_{45}}\right)}{s'_{12}v_{15}}
\end{aligned}
$$

QCD p5l4m function

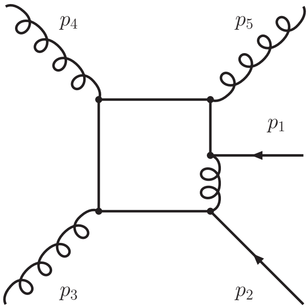


Figure: 4 massive internal lines

Kinematics:

$$(p_1 + p_2)^2 = s_{23} \qquad p_1 p_5 = v_{15}/2 \sim E_5$$

$$(p_3 + p_4)^2 = s_{34} \qquad p_4 p_5 = v_{45}/2 \sim E_5$$

$$p_2 p_3 = s'_{23}/2$$

MB repr QCD p5l4m

Initially 4-d MB representation.
Continuation gives 6 integrals: one 4d, three 3d, one 2d and one 1d.

MB representation of IR part:

$$I_{\mathsf{p5l14m(IR)}} = s_{23}^{-1} s_{12}^{-1} (m^2)^{\epsilon} (-v_{15})^{-2\epsilon-1} \int dz_3 (m^2)^{-z_3} (-s_{12})^{z_3} \times$$
$$\times \frac{\Gamma(-2\epsilon)\Gamma(1+2\epsilon)\Gamma(1-\epsilon+z_3)\Gamma(1-2\epsilon+z_3)\Gamma(-z_3)\Gamma(1+z_3)}{\Gamma(1-2\epsilon)\Gamma(2-2\epsilon+2z_3)}$$

Hypergeometric representation of IR part, $u = m^2/s_{12}$

$$I_{\mathsf{p5l14m(IR)}} = s_{23}^{-1} s_{12}^{-1} (m^2)^{\epsilon} (-v_{15})^{-2\epsilon-1} \Gamma(-2\epsilon)\Gamma(1+2\epsilon) \times$$
$$\times \Gamma(1-\epsilon)\ {}_2F_1\left(1, 1-2\epsilon; 3/2-\epsilon; 1/(4u)\right)/\Gamma(2-2\epsilon)$$

Inverse binomial sums

$$I_{\mathsf{p5l14m(IR)}} = s_{23}^{-1} s_{12}^{-1} (m^2)^{\epsilon} (-v_{15})^{-2\epsilon-1} \times$$
$$\times \frac{1}{2} \int dz_3 (m^2)^{-z_3} (-s_{12})^{z_3} \times$$
$$\times \Gamma(-z_3)\Gamma(z_3+1)^3 \Big[-1/\epsilon + \gamma + 3\psi(1+z_3) - 2\psi(2+2z_3) \Big]$$

Analytical result

$$I_{\mathsf{p5l4m(IR)}} = s_{23}^{-1} s_{12}^{-1} (m^2)^{\epsilon} (-v_{15})^{-2\epsilon-1} \left(-\frac{J_{-1}}{\epsilon} + J_0 \right)$$
$$J_{-1} = \sum_{n=1}^{\infty} \frac{u^{-n}}{\binom{2n}{n}} \frac{1}{n}$$
$$J_0 = \sum_{n=1}^{\infty} \frac{u^{-n}}{\binom{2n}{n}} \frac{1}{n} \left(-S_1(2n-1) + 3S_1(n-1) \right)$$

Inverse binomial sums

$$I_{\mathsf{p5l14m(IR)}} = s_{23}^{-1} s_{12}^{-1} (m^2)^{\epsilon} (-v_{15})^{-2\epsilon-1} \times$$
$$\times \frac{1}{2} \int dz_3 (m^2)^{-z_3} (-s_{12})^{z_3} \times$$
$$\times \Gamma(-z_3)\Gamma(z_3+1)^3 \Big[-1/\epsilon + \gamma + 3\psi(1+z_3) - 2\psi(2+2z_3) \Big]$$

Inverse multiple binomial sums.

$$I_{\mathsf{p5l4m(IR)}} = s_{23}^{-1} s_{12}^{-1} (m^2)^{\epsilon} (-v_{15})^{-2\epsilon-1} \left(-\frac{J_{-1}}{\epsilon} + J_0 \right)$$

$$J_{-1} = \frac{1-y}{1+y} \ln(y), \qquad y = \frac{1-\sqrt{-s_{12}/(4m^2-s_{12})}}{1+\sqrt{-s_{12}/(4m^2-s_{12})}}$$

$$J_0 = \frac{1-y}{1+y} \Big(\ln(y)^2 - 2\ln(1-y)\ln(y) - 2\ln(1+y)\ln(y) - 2\mathsf{Li}_2(-y) - 2\mathsf{Li}_2(y) + \zeta_2 \Big)$$

Inverse Binomial sums

Inverse multiple binomial sum of weight
$W = J + 1 \cdot a1 + 2 \cdot a2 + \cdots + N \cdot aN + 1 \cdot b1 + 2 \cdot b2 + \cdots + M \cdot bM$

$$\mathrm{InvBin}(u)_W(J, \{a1, \ldots, aN\}, \{b1, \ldots, bM\}) =$$
$$= \sum_{n=0}^{\infty} \frac{u^n}{\binom{2n}{n}} \frac{1}{n^J} S_1(n)^{a1} S_2(n)^{a2} \ldots S_N(n)^{aN} \times$$
$$\times S_1(2n)^{b1} S_2(2n)^{b2} \ldots S_M(2n)^{bM}$$

where $S_a(N) = \sum_{n=1}^{\infty} \frac{1}{n^a}$

Binomial sums

Multiple binomial sum of weight
$W = J + 1 \cdot a1 + 2 \cdot a2 + \cdots + N \cdot aN + 1 \cdot b1 + 2 \cdot b2 + \cdots + M \cdot bM$

$$\text{Bin}(u)_W(J, \{a1, \ldots, aN\}, \{b1, \ldots, bM\}) =$$
$$= \sum_{n=0}^{\infty} \binom{2n}{n} u^n \frac{1}{n^J} S_1(n)^{a1} S_2(n)^{a2} \ldots S_N(n)^{aN} \times$$
$$\times S_1(2n)^{b1} S_2(2n)^{b2} \ldots S_M(2n)^{bM}$$

where $S_a(N) = \sum_{n=1}^{\infty} \frac{1}{n^a}$

HPL Basis

For many kind of sums basis functions are known:

$$\mathsf{InvBin}(u)_W(J,\bar{a},0) = \frac{1-y}{1+y}\sum_{\bar{r}_W} C_{r_W}\,\mathsf{HPL}_{(r_1,r_2,\ldots,r_n)}(y)$$

$$\mathsf{Bin}(u)_W(J,\bar{a},0) = \sum_{\bar{r}_W} C_{r_W}\,\mathsf{HPL}_{(r_1,r_2,\ldots,r_n)}(\chi)$$

C_{r_W} - rational coefficients

Common structures involving HPLs:

$$\mathsf{HPL}_{(r_1,r_2,\ldots,r_n)}(y)$$

$$\frac{1}{1+y}\mathsf{HPL}_{(r_1,r_2,\ldots,r_n)}(y)$$

$$\frac{y}{1+y}\mathsf{HPL}_{(r_1,r_2,\ldots,r_n)}(y)$$

$$\frac{1}{(1+y)^2}\mathsf{HPL}_{(r_1,r_2,\ldots,r_n)}(y)$$

$\ldots$

Evaluation with PSLQ

"Experimental" methodology: Integer relation algorithm

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

x_n - real numbers, a_n - integer coefficients.
PSQL algorithm finds a relation or boundary estimate for coefficients.
Needs extremely high precision!!!

$$\text{(Inv)Bin}(X)_W(J, \bar{a}, \bar{b}) = \sum_{n=1}^{N} C_N \, \mathsf{F}_N^{(W)}(X)$$

X - arbitrary real number

Example

$S = \mathsf{Bin}(u)_3(0,\{0,0,1\},0)$

$$S_{\mathsf{sum}} = \sum_{n=1}^{\infty} \frac{u^n}{\binom{2n}{n}} S_3(n)$$

$$S_{\mathsf{int}} = \int_{-3/2} z_3 \frac{(-u)^{-1-z_3}\Gamma(-1-2z_3)\Gamma(1+z_3)\psi^{(2)}(-z_3)}{\Gamma(-z_3)}$$

PSQL result (working precision 1500 digits):
Iteration 7674 eps value=10^{-1461}
Relation= {1,0,0,0,2,-4,0,0,0,0,-4,8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}

$$S = \frac{2}{\chi-1}\Big(\chi \mathsf{Li}_3(-\chi) + \mathsf{Li}_3(-\chi) - 2\chi \mathsf{Li}_3(\chi+1) - 2\mathsf{Li}_3(\chi+1) + 2\chi \mathsf{Li}_2(\chi+1)\log(\chi+1) + 2\mathsf{Li}_2(\chi+1)\log(\chi+1) + 2\zeta(3)\chi + 2\zeta(3) + \chi\log(-\chi)\log^2(\chi+1) + \log(-\chi)\log^2(\chi+1)\Big)$$

Checks

$$S = \mathsf{Bin}(u)_3(0, \{0, 0, 1\}, 0)$$

$$S_{\mathsf{sum}} = \sum_{n=1}^{\infty} \frac{u^n}{\binom{2n}{n}} S_3(n)$$

$$S_{\mathsf{int}} = \int_{-3/2} z_3 \frac{(-u)^{-1-z_3}\Gamma(-1-2z_3)\Gamma(1+z_3)\psi^{(2)}(-z_3)}{\Gamma(-z_3)}$$

Mellin-Barnes, Numerical Summation, PSQL (for $X = 1/9$)

$$-0.104133 - 4.43571 * 10^{-6} I,$$

$$-0.1041351293929038304537440121691219509690,$$

$$-0.1041351293929038304537440121691219509690 + 10^{-52} I,$$

Conclusions

- Mellin-Barnes method is useful for extraction of IR pieces of 1-loop massive n-point functions.
- High level of automatization: *AMBRE*, *MB.m*, *HypExp2*.
- There is still room for improvement (e.g. better continuation procedure, automatic derivation of residua sums, etc).
- Numerical evaluation of residua sums with PSQL.
- Applied to calculation of IR-parts of some LHC-relevant 5- and 6-point functions.
- Extension to two loops is straightforward.

Thank you for your attention!