Minimal Surfaces in AdS/CFT

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$\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ is an interesting theory:

- lots of symmetry (supersymmetry, conformal symmetry, ...)
- theory is UV finite in all orders of perturbation theory
- model theory for QCD
- non perturbative results for Wilson loops and scattering amplitudes via AdS/CFT at **strong** coupling

The last point involves minimal surfaces.

timeline

some great achievements in this field include:

- duality between Wilson loops at strong coupling and minimal surfaces in $AdS \times S$
  [arXiv:9803002, J. Maldacena]
- BDS conjecture for planar MHV gluon amplitudes at strong coupling
  [arXiv:0505205, Z. Bern, L. Dixon, V. Smirnov]
- duality between planar MHV gluon amplitudes at strong coupling and minimal surfaces in AdS $\Rightarrow$ test of BDS conjecture $\Rightarrow$ remainder function
  [arXiv:0705.0303, L.F. Alday, J. Maldacena]
- perturbative calculation of the remainder function at 6 legs and 2 loops
  [arXiv:0911.5332, V. Del Duca, C. Duhr, V. A. Smirnov]
  [arXiv:1006.5703, A. B. Goncharov, M. Spradlin, C. Vergu, A. Volovich]

the AdS/CFT correspondence

AdS/CFT relates type IIB string theory on $AdS_5 \times S^5$ to $\mathcal{N} = 4$ SYM.

$\mathcal{N} = 4$ SYM	Strings on $AdS_5 \times S^5$
't Hooft coupling $\lambda = g_{YM}^2 N$	string tension $T = \frac{\sqrt{\lambda}}{2\pi}$
number of colors N	string coupling $g_s = \frac{\lambda}{4\pi N}$
local operators	string states
scaling dimension of an operator	Energy of the string state
large N (planar) limit ($\lambda = \text{const.}$)	free strings
strong coupling ($\lambda \gg 1$)	classical strings
Wilson loops at strong coupling	minimal surfaces
MHV gluon amplitudes at strong coupling	minimal surfaces

Wilson loops at strong coupling, minimal surfaces

The Wilson loop expectation value is defined as

$$W(C) = \frac{1}{N}\langle 0|\mathrm{Tr}\ \mathrm{P}\ \exp\left(ig\oint_C ds[\dot{x}^\mu A_\mu + |\dot{x}|\theta^I\phi_I]\right)|0\rangle \qquad (1)$$

conjectrue by Maldacena (Wilson loop - minimal surface correspondence):

$$W(C) = \int_{\partial X = C} \mathcal{D}X \exp(-\sqrt{\lambda}S[X])\ , \qquad (2)$$

which for large λ is dominated by $\exp(-\sqrt{\lambda}A)$.
[arXiv:9803002, J. Maldacena]

gluon scattering amplitudes at strong coupling

In the **strong coupling** limit ($\lambda \gg 1$) planar MHV gluon scattering amplitudes correspond to space-like surfaces in AdS_5 with closed light-like polygonal boundary ($\mathcal{A} \propto e^{-\frac{\sqrt{\lambda}}{2}\text{Area}}$).
[arXiv:0705.0303, L.F. Alday, J. Maldacena]

The boundary corresponds to the momentum configuration!

$\Rightarrow$ implies correspondence of Wilson loops and amplitudes at strong coupling. Surprisingly this seems to be valid at weak coupling as well!

- the **only explicitely** known solution corresponds to $\mathcal{A}_4^{MHV}$.
- progress in calculating the regularized area **without** knowing the corresponding surface (Y- System, Thermodynamic Bethe Ansatz)
  [arXiv:0904.0663, L.F. Alday, J. Maldacena]
  [arXiv:0911.4708, L.F. Alday, D. Gaiotto, J. Maldacena]
  [arXiv:1002.2459, L.F. Alday, J. Maldacena, A. Sever, P. Vieira]

$\Rightarrow$ **space-like minimal surfaces in $AdS_5(\times S^5)$ with closed light-like polygonal boundary are interesing!**

Anti-De-Sitter space

Definition of AdS_n via embedding in $\mathbb{R}^{(2,n-1)}$

$$X^i X_i = -(X^0)^2 - (X^{0'})^2 + (X^1)^2 + \cdots + (X^{n-1})^2 = -1 \;, \qquad (3)$$

with the metric $\eta = \mathrm{diag}(-1, -1, 1, \ldots, 1)$.

- AdS_n is a maximally symmetric Lorentzian manifold
- AdS_n has constant negative scalar curvature $R = -n(n-1)$ and constant sectional curvature -1
- it is the homogeneous space $\mathrm{O}(2, n-1) \diagup \mathrm{O}(1, n-1)$
- the isometry group is $\mathrm{O}(2, n-1)$

AdS_n admits a **conformal** boundary.

conformal maps

Let (M, g_M) and (N, g_N) be manifolds with metrics. A map $f : M \to N$ is called **conformal** if $f^* g_N = e^\alpha \, g_M$.

$$AdS_n \overset{\text{conf.}}{\equiv} S^1 \times B_{n-1} \ . \tag{4}$$

$\Rightarrow$ introduce the **conformal boundary** of AdS_n

$$\partial AdS_n = S^1 \times S^{n-2} \ . \tag{5}$$

AdS_n does not have a boundary $\Rightarrow$ the conformal factor will **conformally diverge** when approaching the conformal boundary

Poincaré coordinates

Furthermore $\frac{1}{2}(S^1 \times S^{n-2}) \stackrel{\text{conf.}}{\equiv} \mathbb{R}^{1,n-2}$. One can introduce **Poincaré coordinates** for one half of AdS_n via

$$X^\mu = \frac{x^\mu}{r} \ , \ X^{0'} + X^{n-1} = \frac{1}{r} \ , \ X^{0'} - X^{n-1} = \frac{-x_0^2 + x_1^2 + \cdots + x_{n-2}^2 + r^2}{r} \tag{6}$$

with $\mu \in \{0, 1, \ldots, n-2\}$. One approaches the boundary for $r \to 0$. The metric in these coordinates is

$$g_{\alpha\beta} = \frac{1}{r^2}\mathrm{diag}(-1, 1, \ldots, 1) \ , \tag{7}$$

which is conformal to the flat metric and gives the (n-1) dimensional Minkowski space on the boundary.

conformal group

The **conformal group** is the set of transformations such that the induced metric only differs by a conformal factor.

- isometry group is a subgroup
- conformal group of $\mathbb{R}^{(1,n-2)}$ is $\mathrm{O}(2, n-1)$

$\mathrm{O}(2, n-1)$ **is also the isometry group of** AdS_n**!** The action of this group induces a the conformal group on the boundary.

$$\mathrm{Iso}(AdS_n) = \mathrm{Conf.}(\partial AdS_n) = \mathrm{Conf.}(\mathbb{R}^{(1,n-2)}) \tag{8}$$

Isometry invariance of the string theory is related to conformal invariance of the boundary theory.

The conformal algebra of n dimensional Minkowski space is generated by: n **translations**, $\frac{n(n-1)}{2}$ **Lorentz-boosts** (rotations), 1 **dilatation** and n **special conformal transformations**.

minimal surfaces, breaking of conformal symmetry

- O(2, 4) acts on a light-like boundary configuration
- How big is the space of all possible boundary configurations with n cusps?
- Can one obtain a general configuration out of a special one?
- $\Rightarrow$ a boundary configuration has **3(n-5)** conformally invariant cross-ratios

$\Rightarrow$ for $n = 4, 5$ cusps one just needs one special solution.

Is the area (amplitude) for (4,5) cusps independent of the kinematics?

No, the area needs to be regulated and the **regularization scheme breaks conformal invariance** and introduces a dependence on the Mandelstam variables!

vacuum solutions in $AdS_3 \times S^3$

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why vacuum solutions in $AdS_3 \times S^3$?

We call a solution in $AdS_3 \times S^3$ **vacuum solution** if the solution if it admits a constant induced metric on both factors.

- the string dual lives in $AdS_5 \times S^5 \Rightarrow$ it is natural to examine the problem in the product space
- the tetragon solution is a vacuum solution
- space-like minimal surfaces in the product space can be space-like, time-like and also carry a degenerate metric on the AdS projection $\Rightarrow$ we expect several interesting classes of solutions
- these solutions have not been examined before and can be calculated explicitely

We examine these **vacuum solutions** in

[arXiv:0912.3829, H. Dorn, N. Drukker, G. Jorjadze, C. Kalousios]

[arXiv:0912.3829, H. Dorn, G. Jorjadze, C. Kalousios, L. Megrelidze, S.W.]

Pohlmeyer reduction for AdS projection

A convenient tool for those minimal surfaces is **Pohlmeyer Reduction**.

- introduce an orthonormal reper $\mathcal{E} = \{Y, \frac{\partial_\sigma Y}{|\partial_\sigma Y|}, \frac{\partial_\tau Y}{|\partial_\tau Y|}, N\}$
- rewrite the differential equations using a kind of vielbein formalism
- solve a system of linear differential equations for the matrix $\mathcal{E}$:

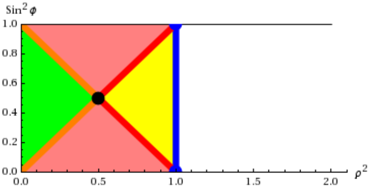
$$\partial_\sigma \mathcal{E} = \mathcal{A}_\sigma \mathcal{E} \ , \quad \partial_\tau \mathcal{E} = \mathcal{A}_\tau \mathcal{E} \ , \quad [\mathcal{A}_\sigma, \mathcal{A}_\tau] + \partial_\tau \mathcal{A}_\sigma - \partial_\sigma \mathcal{A}_\tau = 0 \qquad (9)$$

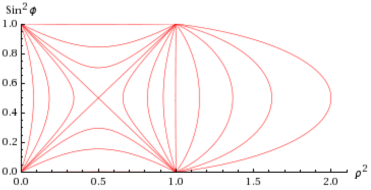
vacuum solutions $\Leftrightarrow \mathcal{A}_\sigma \ , \ \ \mathcal{A}_\tau \ = \text{const.}$

$\Rightarrow$ the integration breaks down to the computation of an exponential

$\mathcal{A}_\sigma$ and $\mathcal{A}_\tau$ depend on to parameters: ρ which paramtereizes the induced metric and ϕ which is the phase of the second fundamental form.

AdS- projection



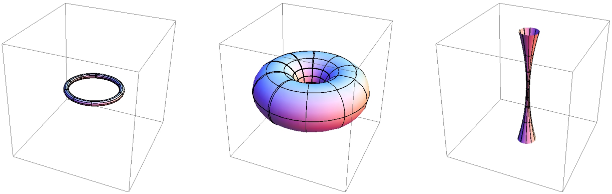


- the AdS projection depends on two parameters $\{\phi, \rho\}$
- the AdS projection is time-like for $\rho^2 < 1$ and space-like for $\rho^2 > 1$
- every class of solutions is represented by a different color in the picture
- the lines represent different solutions that have the same AdS projection

All surfaces have a constant mean curvature in AdS_3.

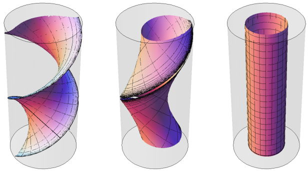
S- projection

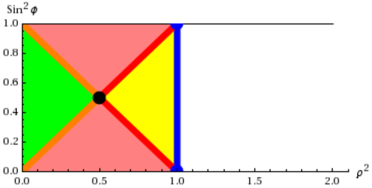
The spherical projections depend in a similar way on two parameters $\{\rho_s, \phi_s\}$. They are tori in S^3 with constant mean curvature.



time-like spinning solutions

There are three time-like (on AdS) solutions that correspond to the green and the pink area and the orange line.

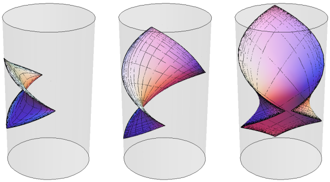


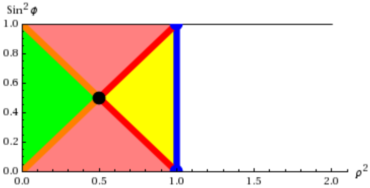


The diameter of the tube and the shape of the solution that touches infinity in two points can be adjusted with a different choice of the parameters $\{\rho, \phi\}$.

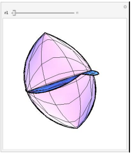
time-like cont.

The depicted solutions belong to the yellow area, the red line and the black dot.



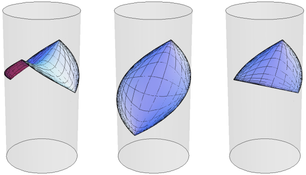


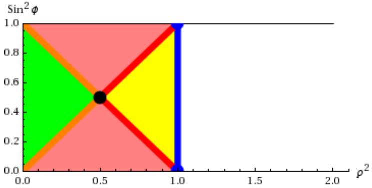
The time-like tetragon can also have different shapes:



spacelike tetragon and degenerate solutions

These solutions correspond to the white area (space-like) tetragon and to the blue line and dots.





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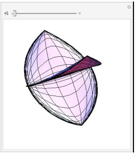
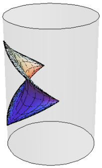


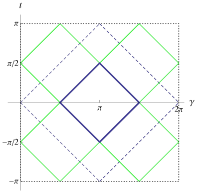
regularized area

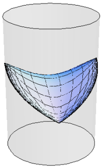
Calculating the area

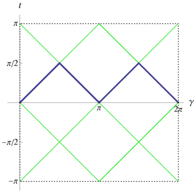
two candidates for amplitudes

From our previous analysis there are two interesting classes:









- time-like tetragon (s- channel): $\cot 2\theta := \frac{\sin\phi\cos\phi}{\rho\sqrt{\rho^2-1}}$, $H = \cot 2\theta$
- space-like tetragon (u- channel): $\tanh 2\theta := \frac{\rho\sqrt{1-\rho^2}}{\sin\phi\cos\phi}$, $H = -\coth 2\theta$

calculation of the area

We choose an isometry transformation that depends on two parameters (a, b) which satisfies the conditions:

- after the boost, the whole configuration is contained in a single Poincaré patch of AdS_4
- the parameters (a, b) allow to adjust the Mandelstam variables s and t

Then calculate a cut-off in AdS_4 which is given via $r_c = \text{const}$. Then we use the full induced metric to calculate the regularized area.

Although the action is invariant under isometry transformation (and thus of s and t), the introduction of a cut-off breaks this conformal invariance!

area for the space-like tetragon solution

The regularized area for the space-like tetragon solution is

$$S_{reg} = \frac{\sqrt{\lambda}}{2\pi} \frac{(\rho^2 + \rho_s^2)}{\rho\sqrt{\rho^2 - 1}} \sin 2\theta \; I(r_c) \; , \tag{10}$$

and

$$\begin{aligned} I(r_c) = \frac{1}{4}\left(\log \frac{r_c^2 \cos^2\theta}{-t4\pi^2}\right)^2 + \frac{1}{4}\left(\log \frac{r_c^2 \sin^2\theta}{-s4\pi^2}\right)^2 \\ - \frac{1}{4}\left(\log \frac{s}{t} \cot^2\theta\right)^2 - \frac{\pi^2}{3} \; , \end{aligned} \tag{11}$$

area for the time-like tetragon

The calculation of the regularized area for a small cutoff r_c in Poincaré coordinates for the time-like tetragon solution leads to

$$S_{reg} = \frac{\sqrt{\lambda}}{2\pi}(\rho^2 + \rho_s^2)\frac{\sinh 2\theta}{\rho\sqrt{1-\rho^2}}\, I(r_c) \ , \tag{12}$$

with

$$\begin{aligned} I(r_c) =& \frac{1}{4}\left(\log\frac{r_c^2\sinh^2\theta}{4\pi^2\, s}\right)^2 + \frac{1}{4}\left(\log\frac{r_c^2\cosh^2\theta}{-4\pi^2\, t}\right)^2 \\ &-\frac{1}{4}\left(\log\frac{s\ \coth^2\theta}{-t}\right)^2 - \frac{\pi^2}{3} \ . \end{aligned} \tag{13}$$

interpretation

The expressions for the time-like and space-like tetragon solutions can be formally related via $\theta \leftrightarrow i\theta$ and continuation to $\rho^2 > 1$.
In order to match with BDS formula the prefactors of the regularized areas have to be equal to 1.

- for the space-like tetragon the prefactor is ≥ 1 and approaches 1 for $\theta \to \frac{\pi}{4}$ and $\rho^2 \to \infty$. $I(r_c)$ then coincides with the pure AdS case. This corresponds to the suppression of the S- projection
- for the time-like tetragon the prefactor is ≥ 2 and approaches its lower bound for $\rho^2 \to 1$ which implies $\theta \to 0$. This causes an additional divergence of $I(r_c)$

These considerations lead to the conclusion that these classes do not have an interpretation as scattering amplitudes at strong coupling.

conclusions / outlook

- on-shell minimal surfaces in $AdS_5(\times S^5)$ are interesting quantities
- we classified an interesting subset of the space-like flat minimal surfaces that make use of the spherical part (which is less explored)
- all projections onto AdS and S have constant mean curvature
- however, the interpretation of these solutions is not yet clear
- provides evidence for neglecting spherical part at strong coupling (solutions may contribute to quantum corrections)
- interesting fact: u-channel configuration can contribute, s-channel configuration not

There are some further interesting subjects

- continue the study of those minimal surfaces
- study the OPE of light-like Wilson loops

Thank you for your attention.