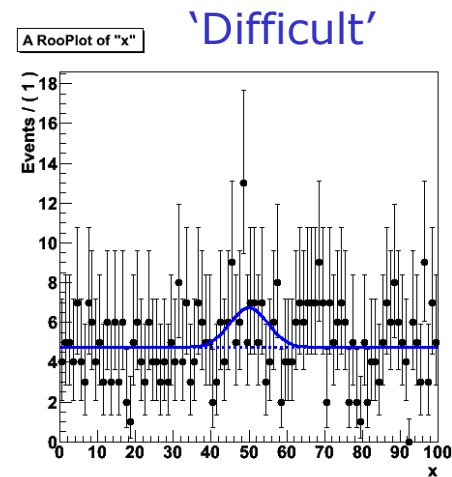
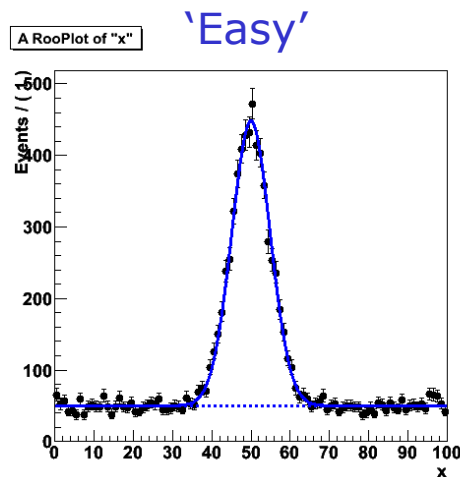


# Confidence intervals Limits & significance

- Null Hypothesis testing – P-values
- Classical or 'frequentist' confidence intervals
- Issues that arise in interpretation of fit result
- Bayesian statistics and intervals

# Introduction

- Issues and differences between methods arise when experimental result contains little information



- Now we focus on the difficult cases
- Most common scenario is establishing the presence of signal in the data (at a certain confidence level), or be able to set limits, in the absence of a convincing signal
  - Connection with hypothesis testing

# Hypothesis testing (reminder)

- Definition of terms
  - Rate of type-I error =  $\alpha$
  - Rate of type-II error =  $\beta$
  - Power of test is  $1-\beta$

		Actual condition	
		Guilty	Not guilty
Decision	Verdict of 'guilty'	True Positive	False Positive (i.e. guilt reported unfairly) <b>Type I error</b>
	Verdict of 'not guilty'	False Negative (i.e. guilt not detected) <b>Type II error</b>	True Negative

- Treat hypotheses asymmetrically
  - Null hypo is special  $\rightarrow$  Fix rate of type-I error
- Now can define a well stated goal
  - Maximize the power of test (minimized rate of type-II error) for given  $\alpha$

# Formulating the question precisely

- When making statistical inference on data samples that contain little information, precise formulation of question and assumption made, become very important
- Let's start with a very basic formulation on the question of discovery.
- Hypothetical case for "SuperSymmetry" discovery
  - Simulation for SM – Predicts 3 events (Poisson,  $\mu$  exactly known)
  - Simulation for SUSY – Predicts 6 events  $\rightarrow$  9 events in total
  - Observed event count in data: 8 events
- How do you conclude (or not) that you've discovered supersymmetry?
  - You expect 9 events (with SUSY), you see 8, looks promising

# Formulating the question precisely

- NB: Proving that you see SUSY is very hard!
  - Usually not the 1<sup>st</sup> question to resolve, instead
- Instead: **Can you prove the SM is wrong?**
  - I.e. what is the probability when expect 3 events we observe, with SM processes only?
  - Note that this question is easier to answer: you don't even need any SUSY simulation to (dis)prove it
- Connection with hypothesis testing
  - $H_0 = \text{SM}$ ,  $H_1 = \text{SM+SUSY}$ ,
  - Aiming to disprove  $H_0 \rightarrow$  care primarily about rate of Type-I error (a)  $\rightarrow$  'False Positive' (claim discovery when there is none)
  - NB: To quantify Type-II error (b)  $\rightarrow$  'False Negative' (claim no discovery when you could have) you *do* need SUSY events
    - This will then also depend on SUSY modeling details & parameters

# Significance, Probability

- How do you formulate your result
  - There is a X% probability that there is no SUSY in nature?
  - If there is no SUSY in nature, Y% of repeated experiments will report an excess of events that observed (or larger)
- Are these statements equivalent?
- Do both statements result in the same numeric value?
  - I.e. is  $Y\% = 100\% - X\%$
- Need to discuss fundamentals of probability and statistics more before proceeding.

# Definition of “Probability”

- Abstract mathematical probability  $P$  can be defined in terms of sets and axioms that  $P$  obeys. If the axioms are true for  $P$ , then  $P$  obeys Bayes' Theorem (see next slides)

$$P(B|A) = P(A|B) P(B) / P(A).$$

- Two established\* incarnations of  $P$  are:
- **1) Frequentist  $P$ :** limiting frequency in ensemble of imagined repeated samples (as usually taught in Q.M.).  
 $P(\text{constant of nature})$  and  $P(\text{SUSY is true})$  do not exist (in a useful way) for this definition of  $P$  (at least in one universe).
- **2) (Subjective) Bayesian  $P$ :** subjective degree of belief. (de Finetti, Savage)  $P(\text{constant of nature})$  and  $P(\text{SUSY is true})$  exist for You. Shown to be basis for coherent personal decision-making.

*\*It is important to be able to work with either definition of  $P$ , and to know which one you are using!*

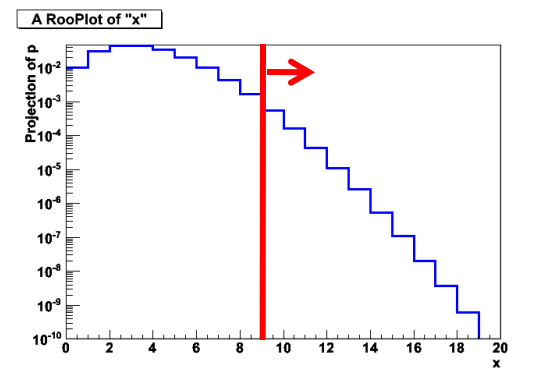
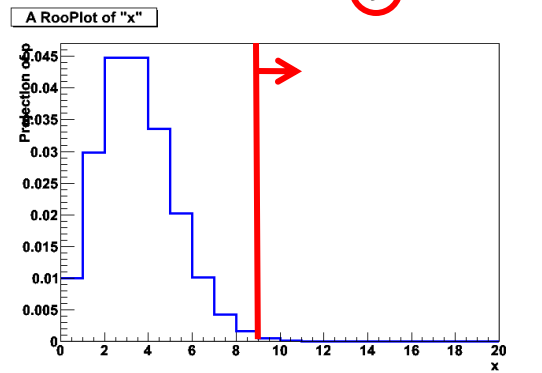
# Frequentist P – the initial example

- Work out initial example (disproving SM)

➡ Prediction N=3  
Measurement N=9

- Can we calculate probability that SM mimics N=9 (i.e. result is a 'false positive')?
  - Calculation details depend on how measurement was done (fit, counting etc..)
  - Simplest case: counting experiment, Poisson process

$$p = \int_9^{\infty} \text{Poisson}(n; \mu = 3) dn = 0.0038 \quad = \text{'p value'}$$





## Frequentist P – working out example #2

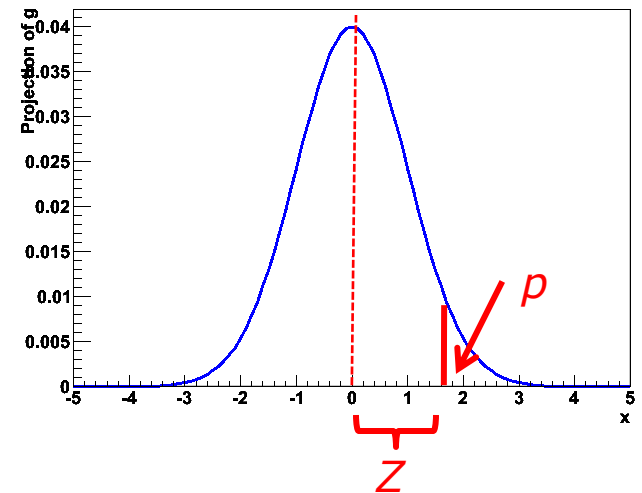
- P-value - If you repeat experiment many times, given fraction of experiments will result in result more extreme than observed value
  - In this example, only 0.38% of experiments will result in an observation of 9 or more events when 3 are expected.
- P-Value vs Z-value (significance)
  - Often defines significance Z as the number of standard deviations that a Gaussian variable would fluctuate in one direction to give the same  $p$ -value.

$$p = \int_Z^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1 - \Phi(Z)$$

TMath::Erfc

$$Z = \Phi^{-1}(1 - p)$$

TMath::NormQuantile



Wouter Verkerke, NIKHEF

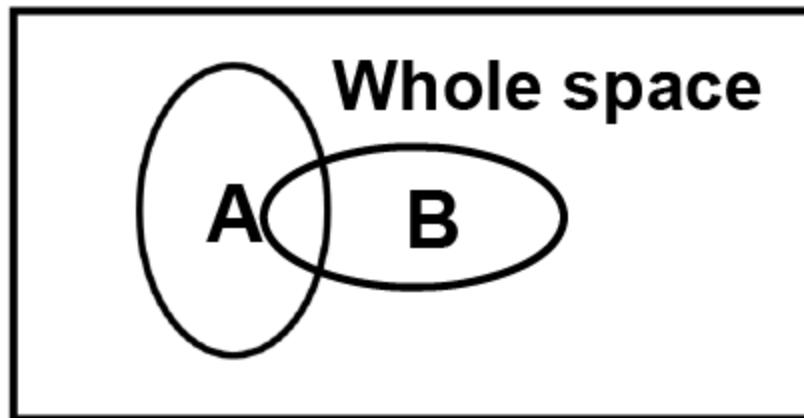
# Bayes Theorem in pictures

- Rev. Thomas Bayes
- 1702 – 7 April 1761
- Bayes Theorem

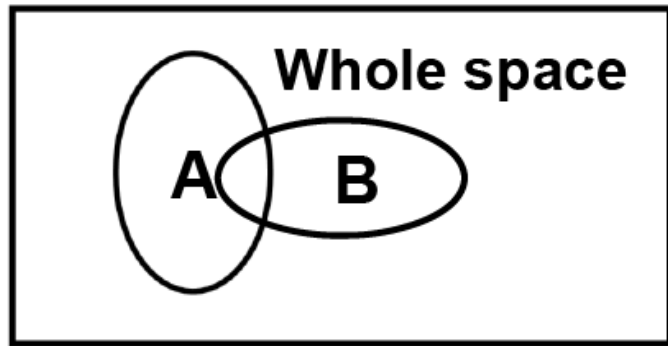
$$P(B|A) = P(A|B) P(B) / P(A).$$



- *Essay "Essay Towards Solving a Problem in the Doctrine of Chances" published in Philosophical Transactions of the Royal Society of London in 1764*



# Bayes' Theorem in Pictures



$$P(A) = \frac{\text{Area of A}}{\text{Area of Whole space}}$$

$$P(B) = \frac{\text{Area of B}}{\text{Area of Whole space}}$$

$$P(A|B) = \frac{\text{Area of A} \cap B}{\text{Area of B}}$$

$$P(B|A) = \frac{\text{Area of A} \cap B}{\text{Area of A}}$$

$$P(A \cap B) = \frac{\text{Area of A} \cap B}{\text{Area of Whole space}}$$

$$P(A) \times P(B|A) = \frac{\text{Area of A}}{\text{Area of Whole space}} \times \frac{\text{Area of A} \cap B}{\text{Area of B}} = \frac{\text{Area of A} \cap B}{\text{Area of Whole space}} = P(A \cap B)$$

$$P(B) \times P(A|B) = \frac{\text{Area of B}}{\text{Area of Whole space}} \times \frac{\text{Area of A} \cap B}{\text{Area of A}} = \frac{\text{Area of A} \cap B}{\text{Area of Whole space}} = P(A \cap B)$$

$$\Rightarrow P(B|A) = P(A|B) \times P(B) / P(A)$$

# What is the “Whole Space”?

- Note that for probabilities to be well-defined, the “whole space” needs to be defined, which in practice introduces assumptions and restrictions.
- Thus the “whole space” itself is more properly thought of as a conditional space, conditional on the assumptions going into the model (Poisson process, whether or not total number of events was fixed, etc.).
- Furthermore, it is widely accepted that restricting the “whole space” to a relevant subspace can sometimes improve the quality of statistical inference –see the discussion of “Conditioning” in later slides.

# Example of Bayes' Theorem Using Frequentist P

- A b-tagging method is developed and one measures:

- $P(\text{btag} | \text{b-jet})$ , i.e., efficiency for tagging b's
- $P(\text{btag} | \text{not a b-jet})$ , i.e., efficiency for background
- $P(\text{no btag} | \text{b-jet}) = 1 - P(\text{btag} | \text{b-jet})$ ,
- $P(\text{no btag} | \text{not a b-jet}) = 1 - P(\text{btag} | \text{not a b-jet})$

- **Question:** Given a selection of jets tagged as b-jets, what fraction of them is b-jets?  
I.e., what is  $P(\text{b-jet} | \text{btag})$  ?

- **Answer:** *Cannot be determined from the given information!*

- Need also:  $P(\text{b-jet})$ , the true fraction of *all jets that are b-jets*.  
Then Bayes' Theorem inverts the conditionality:

$$P(\text{b-jet} | \text{btag}) \propto P(\text{btag} | \text{b-jet}) P(\text{b-jet})$$

# Example of Bayes' Theorem Using Bayesian P

- In a background-free experiment, a theorist uses a “model” to predict a signal with Poisson mean of 3 events. From Poisson formula we know
  - $P(0 \text{ events} \mid \text{model true}) = 3^0 e^{-3} / 0! = 0.05$
  - $P(0 \text{ events} \mid \text{model false}) = 1.0$
  - $P(>0 \text{ events} \mid \text{model true}) = 0.95$
  - $P(>0 \text{ events} \mid \text{model false}) = 0.0$
- The experiment is performed and zero events are observed.
- **Question:** Given the result of the expt, what is the probability that the model is true?

I.e., What is  $P(\text{model true} \mid 0 \text{ events})$  ?

# Example of Bayes' Theorem Using Bayesian P

- **Answer:** *Cannot be determined from the given information!*
  - *Need in addition:  $P(\text{model true})$ , the degree of belief in the model prior to the experiment. Then using Bayes' Thm*
  - $P(\text{model true} \mid 0 \text{ events}) \propto P(0 \text{ events} \mid \text{model true}) P(\text{model true})$
- If “model” is S.M., then still very high degree of belief after experiment!
- If “model” is large extra dimensions, then low prior belief becomes even lower.
  - N.B. Of course this example is over-simplified

## A Note re *Decisions*

- Suppose that as a result of the previous experiment, your degree of belief in the model is  $P(\text{model true} \mid 0 \text{ events}) = 99\%$ , and you need to *decide whether or not to take an action*
  - *making a press release, or planning your next experiment, based on the model being true.*
- **Question: What should you decide?**
- Answer: *Cannot be determined from the given information!*
  - Need in addition: the utility function (or cost function), which gives the relative costs (to You) of a Type I error (declaring model false when it is true) and a Type II error (not declaring model false when it is false).
- Thus, Your *decision*, such as where to invest your time or money, requires two subjective inputs: Your prior probabilities, and the relative costs to You of outcomes.
- Statisticians often focus on decision-making; in HEP, the tradition thus far is to communicate experimental results (well) short of formal decision calculations. *One thing should become clear: classical "hypothesis testing" is not a complete theory of decision-making!*



# At what $p/Z$ value do we claim discovery?

- HEP folklore: claim discovery when  $p$ -value of background only hypothesis is  $2.87 \times 10^{-7}$ , corresponding to significance  $Z = 5$ .
- This is very subjective and really should depend on the prior probability of the phenomenon in question, e.g.,

<u>phenomenon</u>	<u>reasonable <math>p</math>-value for discovery</u>
$D^0\bar{D}^0$ mixing	$\sim 0.05$
Higgs	$\sim 10^{-7}$ (?)
Life on Mars	$\sim 10^{-10}$
Astrology	$\sim 10^{-20}$

- Cost of type-I error (false claim of discovery) can be high
  - Remember cold nuclear fusion 'discovery'

# Bayes' Theorem Generalized to Probability Densities

- Original Bayes Thm:

$$P(B|A) \propto P(A|B) P(B).$$

- Let probability density function  $p(\mathbf{x}|\mu)$  be the conditional pdf for data  $\mathbf{x}$ , given parameter  $\mu$ . Then Bayes' Thm becomes

$$p(\mu|\mathbf{x}) \propto p(\mathbf{x}|\mu) p(\mu).$$

- Substituting in a set of **observed data**,  $x_0$ , and recognizing the likelihood, written as  $L(x_0|\mu)$ ,  $L(\mu)$ , then

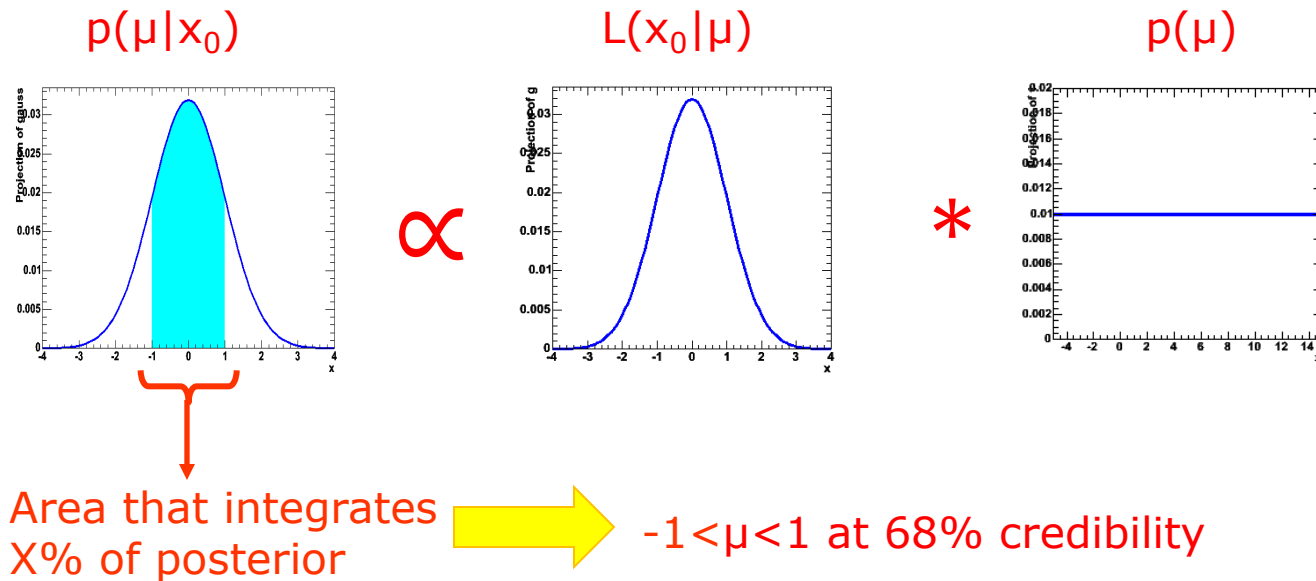
$$p(\mu|x_0) \propto L(x_0|\mu) p(\mu),$$

where:

- $p(\mu|x_0)$  = posterior pdf for  $\mu$ , given the results of this experiment
  - $L(x_0|\mu)$  = Likelihood function of  $\mu$  from the experiment
  - $p(\mu)$  = prior pdf for  $\mu$ , before incorporating the results of this experiment
- Note that there is one (and only one) probability density in  $\mu$  on each side of the equation, again consistent with the likelihood *not* being a density.

# Bayes' Theorem Generalized to pdfs

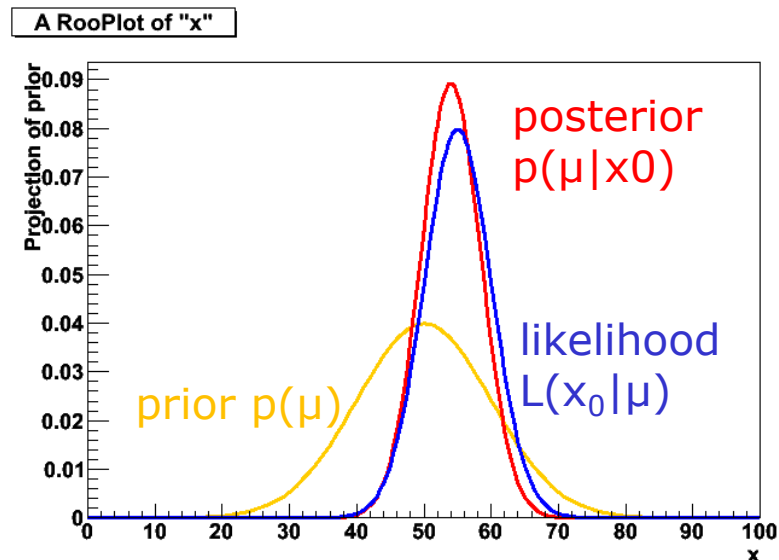
- Graphical illustration of  $p(\mu|x_0) \propto L(x_0|\mu) p(\mu)$



- Upon obtaining  $p(\mu|x_0)$ , the *credibility* of  $\mu$  being in any interval can be calculated by integration.
  - To make a *decision* as to whether or not  $\mu$  is in an interval or not (e.g., whether or not  $\mu > 0$ ), one requires a further subjective input: the cost function (or utility function) for making wrong decisions

# Choosing Priors

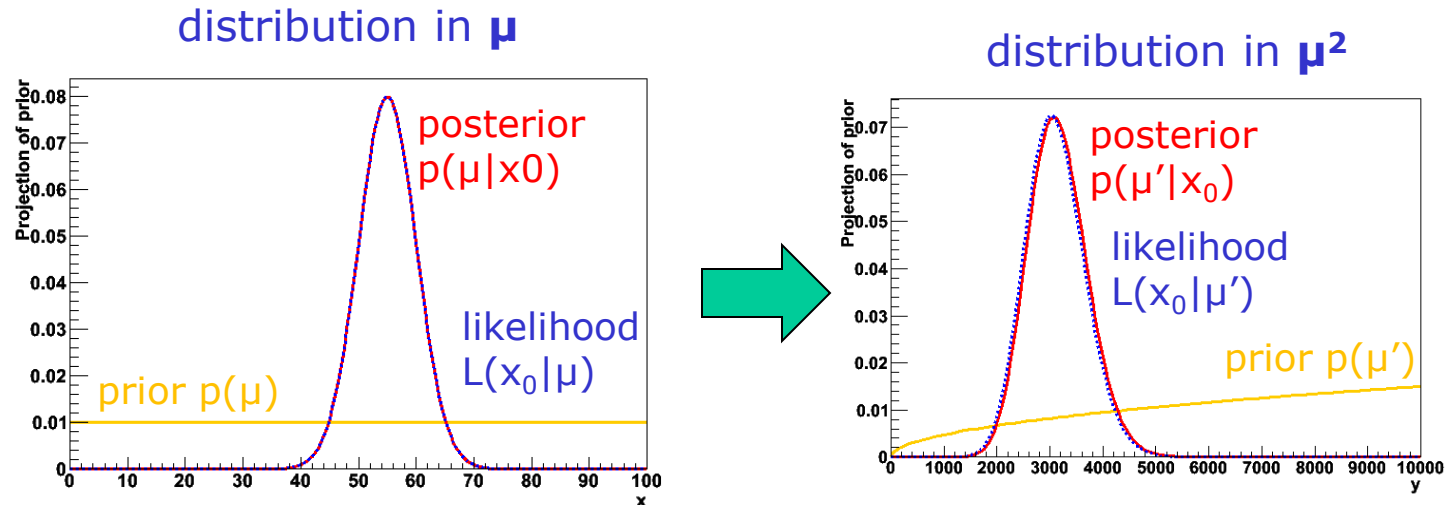
- When using the Bayesian formalism you always have a prior. What should you put in there?
- When there is clear prior knowledge, it is usually straightforward what to choose as prior
  - Example: prior measurement of  $\mu = 50 \pm 10$



- Posterior represents updated belief. But sometimes we only want to publish result of *this* experiment, or there is no prior information. What to do?

# Choosing Priors

- **Common but thoughtless choice:** a flat prior
  - Flat implies choice of metric. Flat in  $x$ , is not flat in  $x^2$



- Flat prior implies choice on given metric
  - Conversely you make any prior flat by a appropriate coordinate transformation (i.e a probability integral transform)
  - 'Preferred metric' has often no clear-cut answer. (E.g. when measuring neutrino-mass-squared, state answer in  $m$  or  $m^2$ )
  - In multiple dimensions even more issues (flat in  $x, y$  or flat in  $r, \phi$ ?)

# Probability Integral Transform

- "...seems likely to be one of the most fruitful conceptions introduced into statistical theory in the last few years" –Egon Pearson (1938)
- Given continuous  $x \in (a,b)$ , and its pdf  $p(x)$ , let

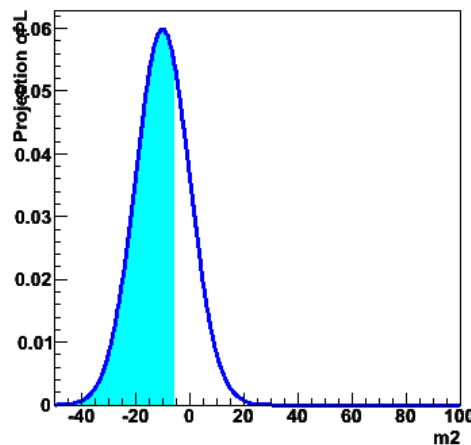
$$y(x) = \int_a^x p(x') dx'.$$

- Then  $y \in (0,1)$  and  $p(y) = 1$  (uniform) for all  $y$ . (!)
- So there *always* exists a metric in which the pdf is uniform.
  - The specification of a Bayesian prior pdf  $p(\mu)$  for parameter  $\mu$  is equivalent to the choice of the metric  $f(\mu)$  in which the pdf is uniform.

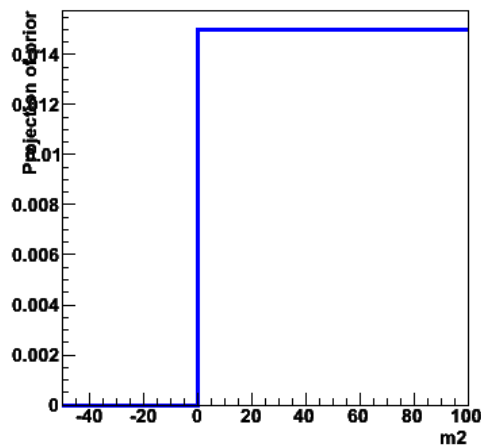
# Using priors to exclude unphysical regions

- Priors provide a simple way to exclude unphysical regions from consideration
- Simplified example situations for a measurement of  $m_\nu^2$ 
  1. Central value comes out negative (= unphysical).
  2. Upper limit (68%) may come out negative, e.g.  $m^2 < -5.3$ , not so clear what to make of that

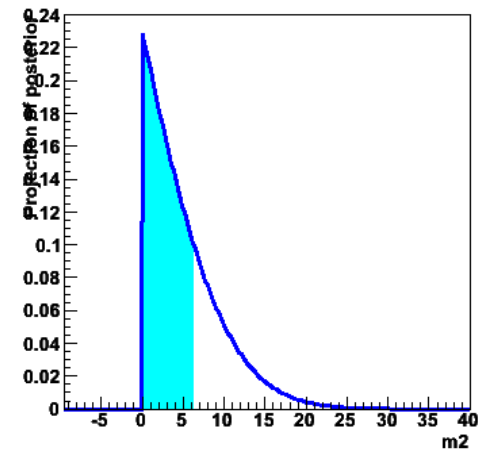
$p(\mu|x_0)$  with flat prior



$p'(\mu)$



$p(\mu|x_0)$  with  $p'(\mu)$



- Introducing prior that excludes unphysical region ensure limit in physical range of observable ( $m^2 < 6.4$ )
- NB: Previous considerations on appropriateness of flat prior for domain  $m^2 > 0$  still apply

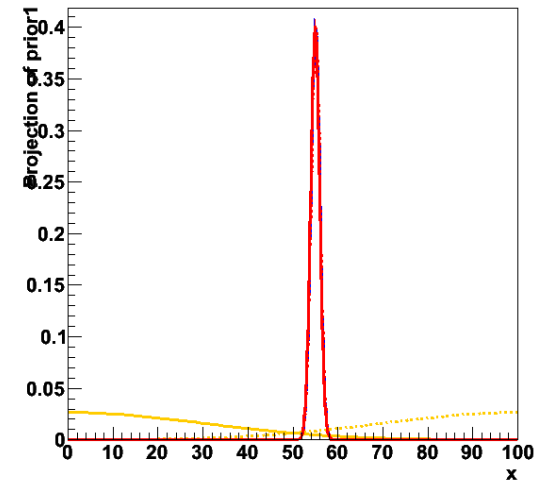
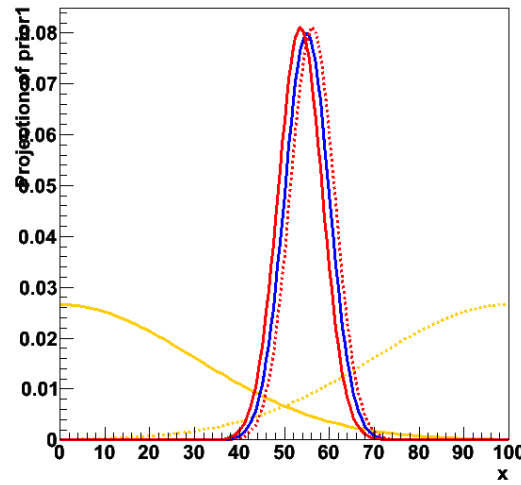
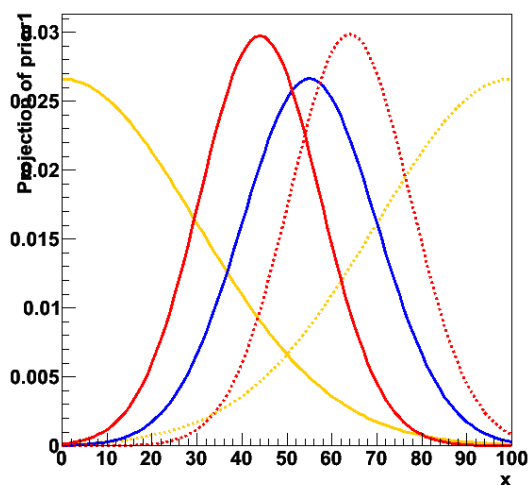
# Non-subjective priors?

- The question is: can the Bayesian formalism be used by scientists to report the results of their experiments in an “objective” way (however one defines “objective”), and does any of the coherence remain when subjective P is replaced by something else?
- *Can one define a prior  $p(\mu)$  which contains as little information as possible, so that the posterior pdf is dominated by the likelihood?*
  - A bright idea, vigorously pursued by physicist Harold Jeffreys in mid-20th century:
  - The really *really* thoughtless idea\*, recognized by Jeffreys as such, but dismayingly common in HEP: just choose  $p(\mu)$  uniform in whatever metric you happen to be using!
- “Jeffreys Prior” answers the question using a prior uniform in a metric related to the Fisher information.
  - Unbounded mean  $\mu$  of gaussian:  $p(\mu) = 1$
  - Poisson signal mean  $\mu$ , no background:  $p(\mu) = 1/\sqrt{\mu}$
- Many ideas and names around on non-subjective priors
  - Objective priors? *Non-informative* priors? *Uninformative* priors?
  - Vague priors? Ignorance priors? Reference priors?
- Kassand & Wasserman who have compiled a list of them, suggest a neutral name : *Priors selected by “formal rules”*.
  - Whatever the name, keep in mind that choice of prior in one metric determines it in all other metrics: be careful in the choice of metric in which it is uniform!
  - N.B. When professional statisticians refer to “flat prior”, they usually mean the Jeffreys prior.



# Sensitivity Analysis

- Since a Bayesian result depends on the prior probabilities, which are either personalistic or with elements of arbitrariness, it is widely recommended by Bayesian statisticians to study the *sensitivity* of the result to varying the prior.
- Sensitivity generally decreases with precision of experiment



- Some level of arbitrariness – what variations to consider in sensitivity analysis

# Bayesian Probability

- Bayesian probability is often the 'natural' framework in which people (& scientists) think.
- If you read " $90 < M(X) < 100$ " to mean that the **true**  $M(X)$  has a 68% probability of being between 90-100 **then you're thinking in terms of Bayesian probability**
- Strictly speaking you're quantifying your belief in  $M(X)$  (or perhaps our 'collective belief as HEP scientists' as true value in nature of  $M(X)$  is fixed (but unknown))
- In the Bayesian framework you always have a prior.
  - If you didn't put one in, you're assuming it to be flat in your current choice of metric

# What Can Be Computed without Using a Prior?

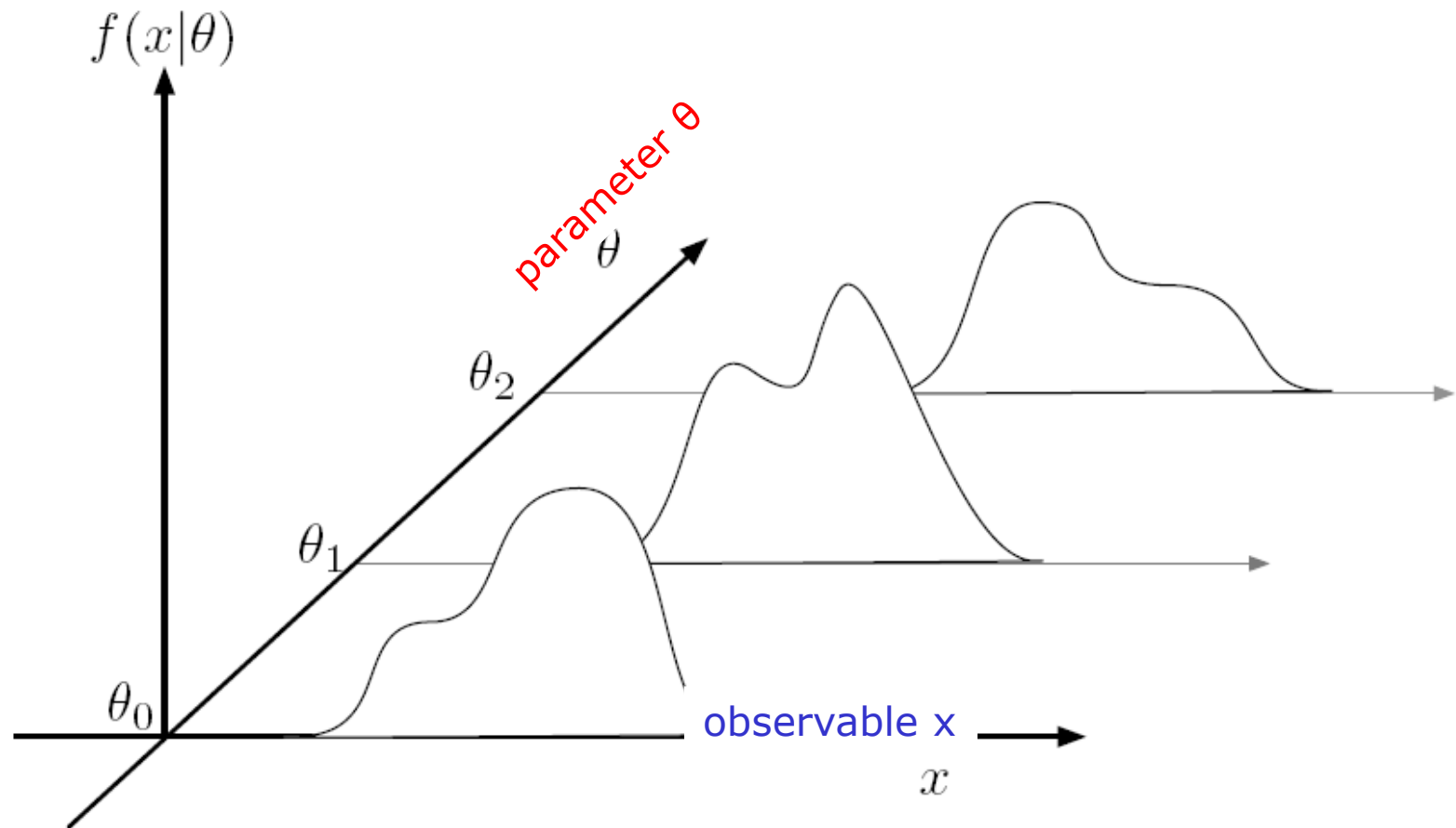
- *Not*  $P(\text{constant of nature} \mid \text{data})$ .
  1. *Confidence Intervals* for parameter values, as defined in the 1930's by Jerzy Neyman.
  2. *Likelihood ratios*, the basis for a large set of techniques for point estimation, interval estimation, and hypothesis testing.
- These can both be constructed using frequentist definition of  $P$ .
- Compare and contrast them with Bayesian methods.

# Confidence Intervals

- “Confidence intervals”, and this phrase to describe them, were invented by Jerzy Neyman in 1934-37.
  - While statisticians mean Neyman’s intervals (or an approximation) when they say “confidence interval”, in HEP the language tends to be a little loose.
  - Recommend using “confidence interval” only to describe intervals corresponding to Neyman’s construction (or good approximations thereof), described below.
- The slides contain the crucial information, but you will want to cycle through them a few times to “take home” how the construction works, since it is really ingenious – perhaps a bit *too* ingenious given how often confidence intervals are misinterpreted.
- In particular, you will understand that the confidence level does *not* tell you “how confident you are that the unknown true value is in the interval” –only a *subjective* Bayesian credible interval has that property!

# How to construct a Neyman Confidence Interval

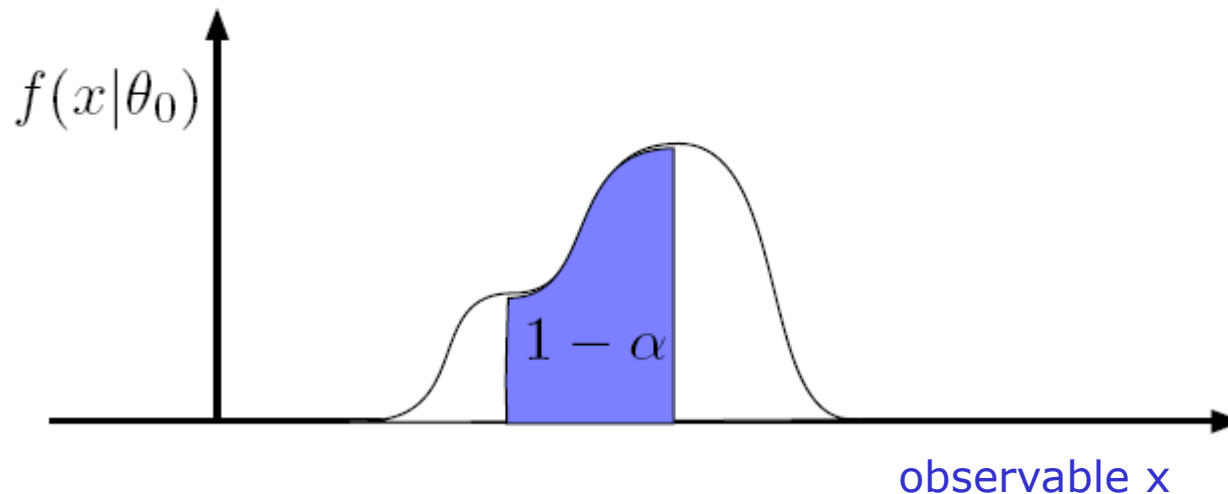
- Simplest experiment: one measurement ( $x$ ), one theory parameter ( $\theta$ )
- For each value of **parameter  $\theta$** , determine distribution in **observable  $x$**



# How to construct a Neyman Confidence Interval

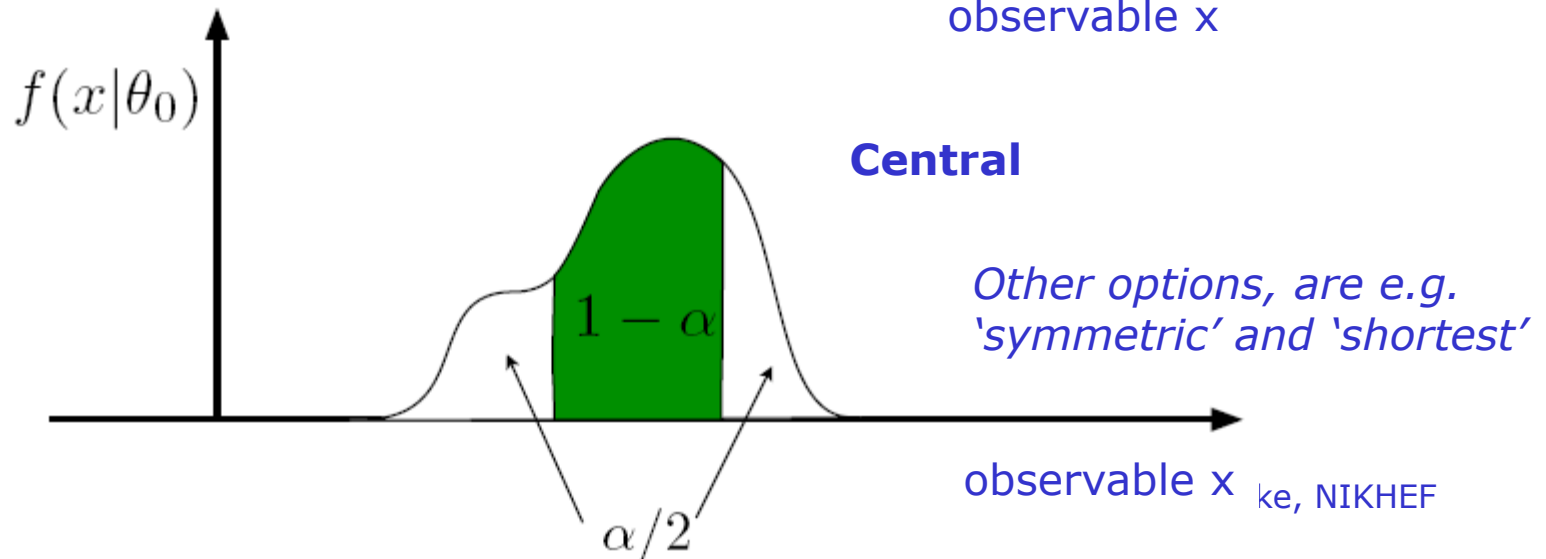
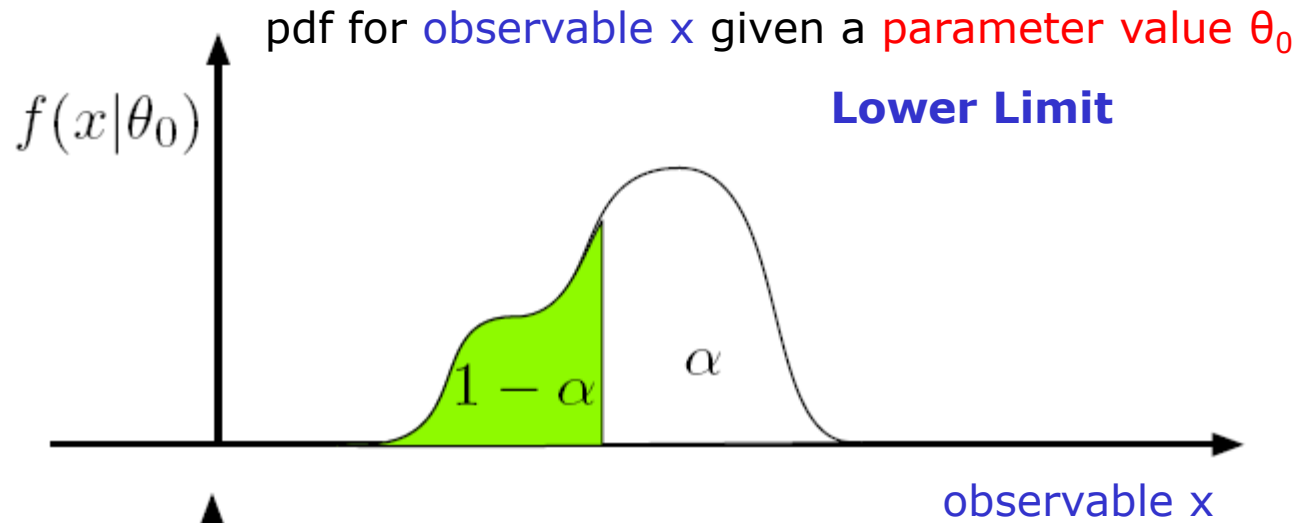
- Focus on a slice in  $\theta$ 
  - For a  $1-\alpha\%$  confidence Interval, define **acceptance interval** that contains  $100\%-\alpha\%$  of the probability

pdf for observable  $x$   
given a parameter value  $\theta_0$



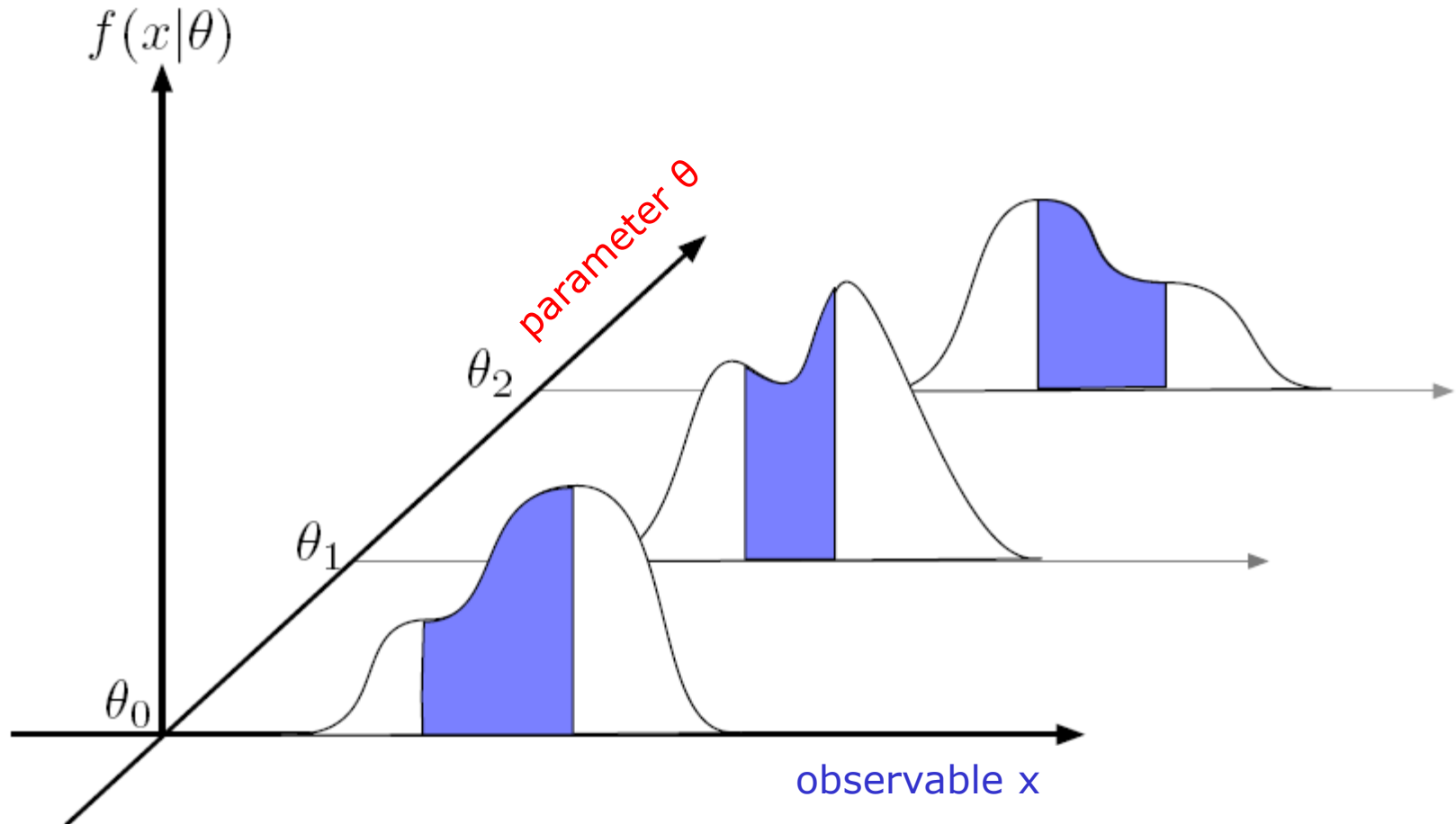
# How to construct a Neyman Confidence Interval

- Definition of acceptance interval is not unique
  - Algorithm to define acceptance interval is called '**ordering rule**'



# How to construct a Neyman Confidence Interval

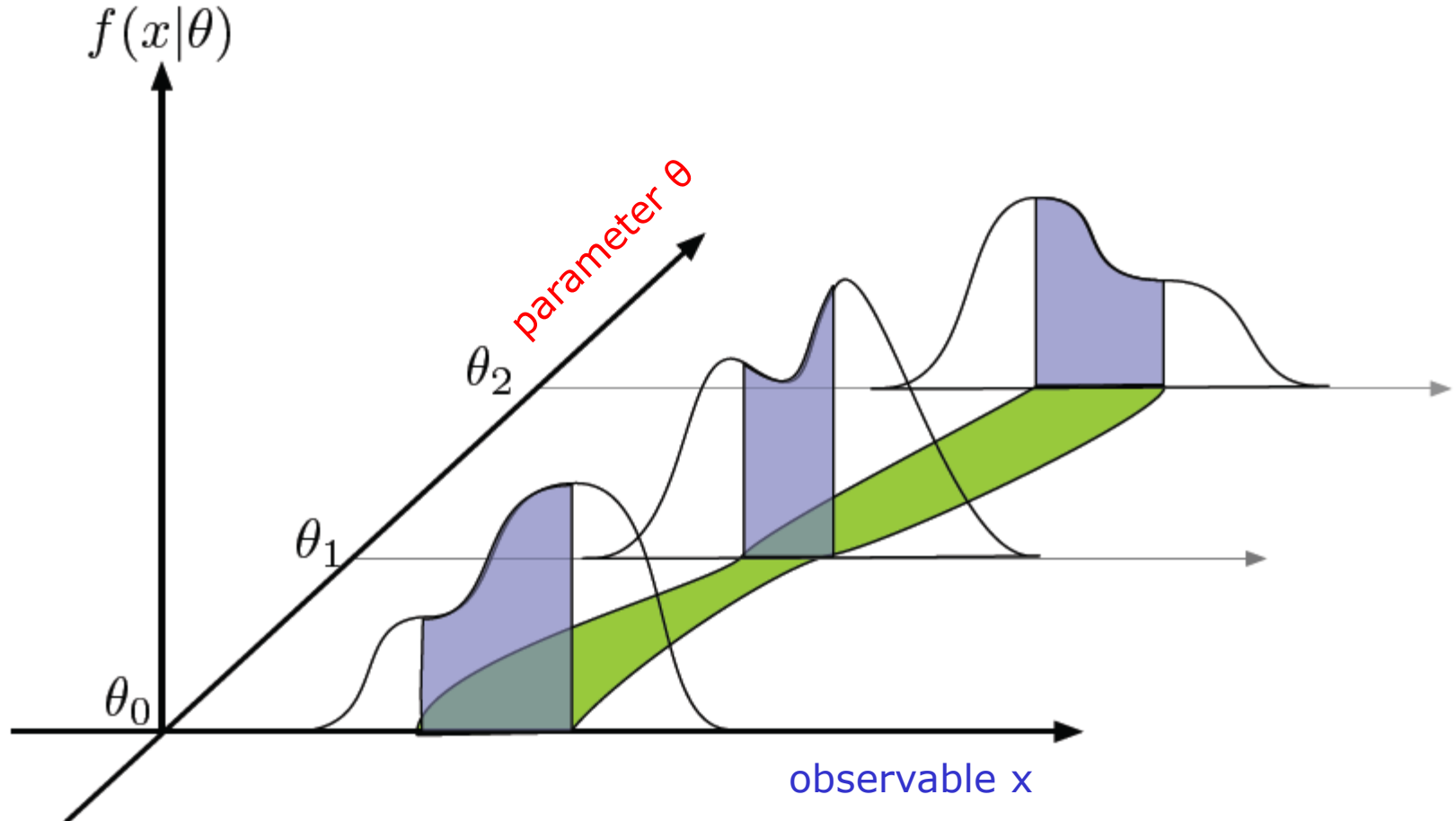
- Now make an acceptance interval in **observable  $x$**  for each value of **parameter  $\theta$**





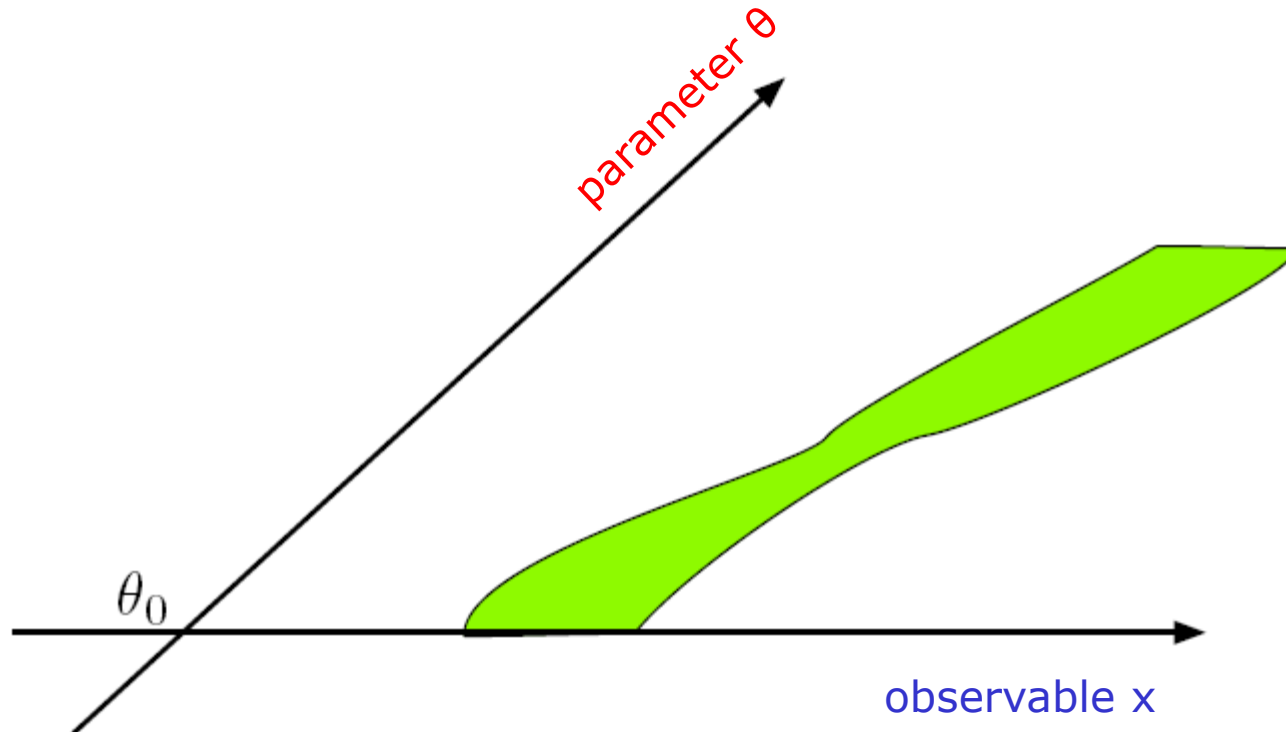
# How to construct a Neyman Confidence Interval

- This makes the confidence belt
  - The region of data in the confidence belt can be considered as consistent with **parameter  $\theta$**



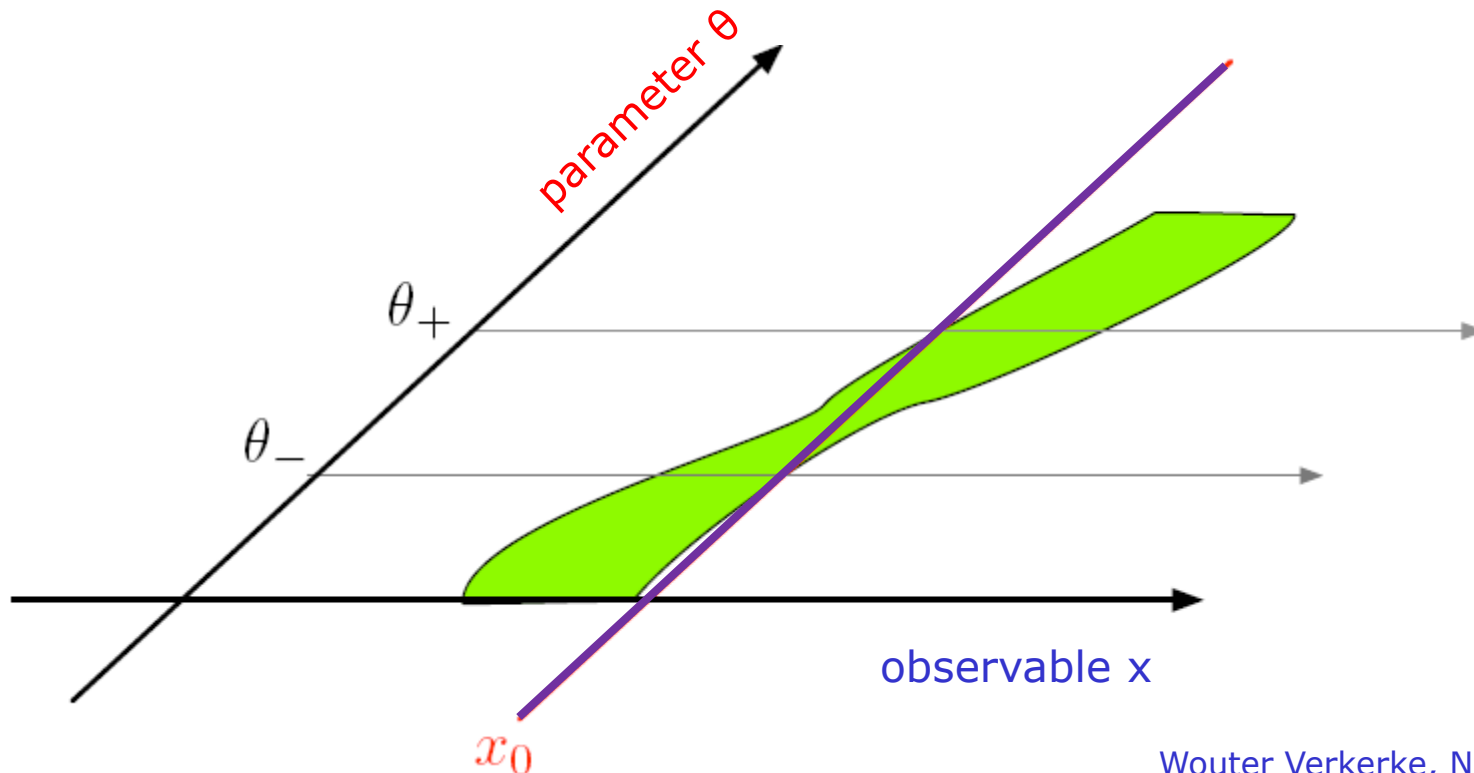
# How to construct a Neyman Confidence Interval

- This makes the confidence belt
  - The region of data in the confidence belt can be considered as consistent with **parameter  $\theta$**



# How to construct a Neyman Confidence Interval

- The confidence belt can be constructed in advance of any measurement, it is a property of the model, not the data
- Given a measurement  $x_0$ , a confidence interval  $[\theta_+, \theta_-]$  can be constructed as follows
- The interval  $[\theta_-, \theta_+]$  has a 68% probability to cover the true value



# Confidence interval – summary

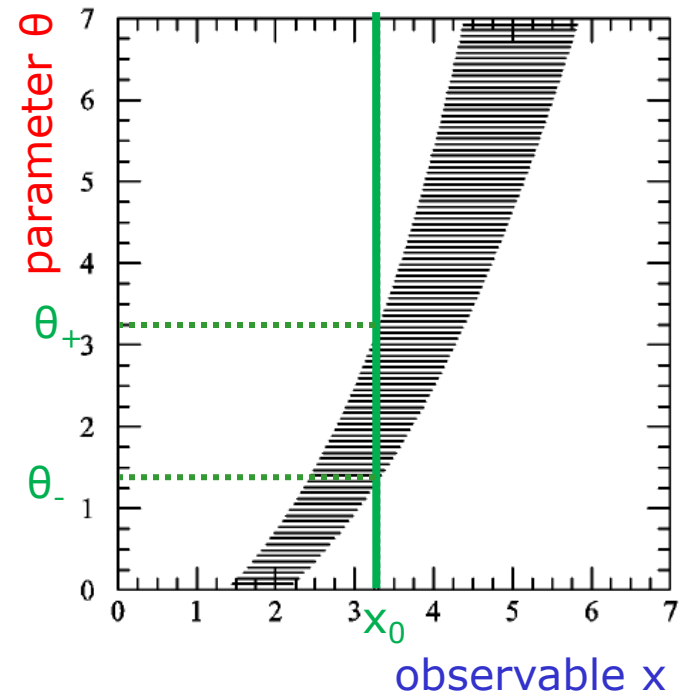
- *Note that this result does NOT amount to a probability density distribution in the true value of  $\theta$*
- Let the unknown true value of  $\theta$  be  $\theta_t$ .

In repeated expt's, the confidence intervals obtained will have different endpoints  $[\theta_1, \theta_2]$ , since the endpoints are functions of the randomly sampled  $x$ .

A little thought will convince you that a fraction C.L. =  $1 - \alpha$  of intervals obtained by Neyman's construction will contain ("cover") the fixed but unknown  $\mu_t$ . i.e.,

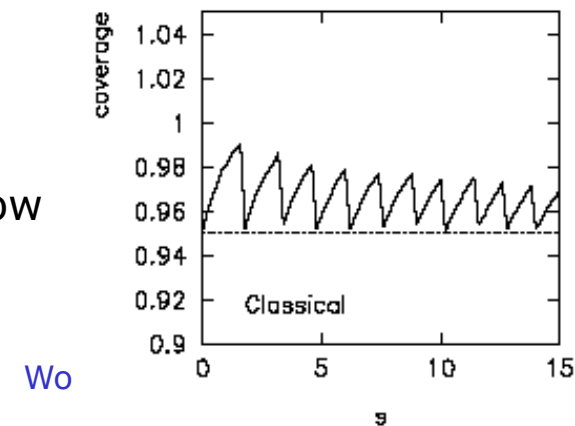
$$P(\theta_t \in [\theta_1, \theta_2]) = \text{C.L.} = 1 - \alpha.$$

- The random variables in this equation are  $\theta_1$  and  $\theta_2$ , and not  $\theta_{tr}$
- Coverage is a property of the set, not of an individual interval!
- It is true that the confidence interval consists of those values of  $\theta$  for which the observed  $x$  is among the most probable to be observed.
  - In precisely the sense defined by the ordering principle used in the Neyman construction



# Coverage

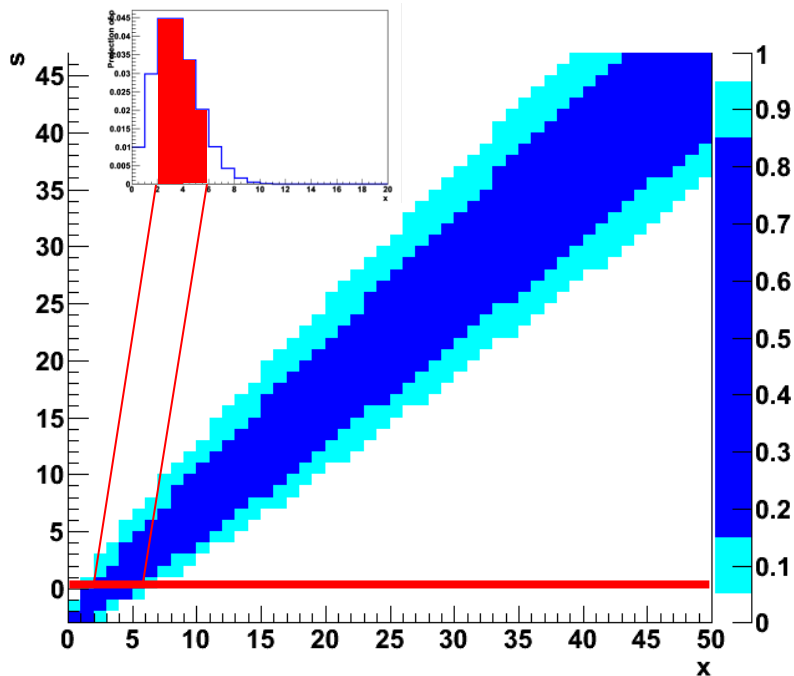
- Coverage = Calibration of confidence interval
  - Interval has coverage if probability of true value in interval is  $\alpha\%$  for all values of  $\mu$
  - It is a property of the procedure, not an individual interval
- Over-coverage : probability to be in interval  $>$  C.L
  - Resulting confidence interval is conservative
- Under-coverage : probability to be in interval  $<$  C.L
  - Resulting confidence interval is optimistic
  - Under-coverage is undesirable  $\rightarrow$  You may claim discovery too early
- Exact coverage is difficult to achieve
  - For Poisson process impossible due to discrete nature of event count
  - “Calibration graph” for preceding example below



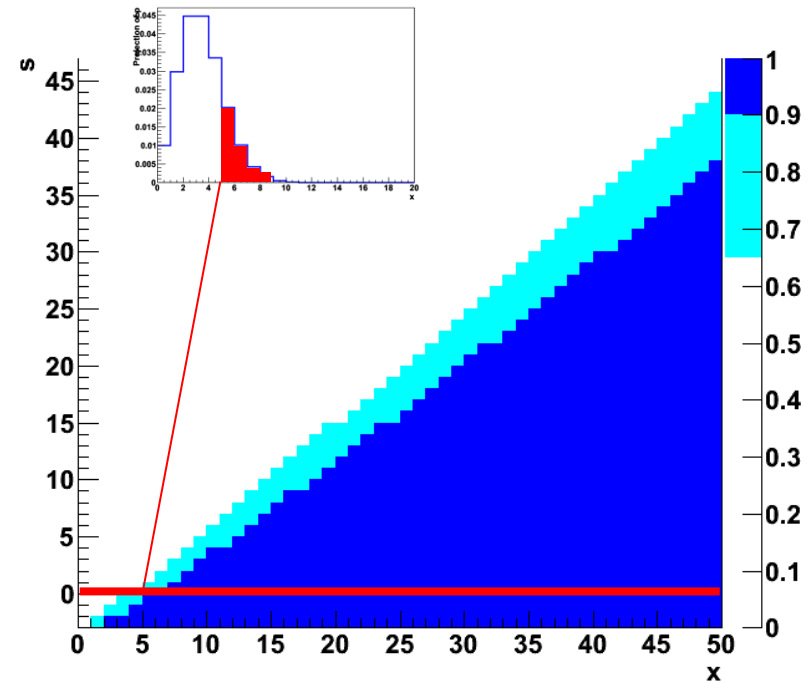
# Confidence intervals for Poisson counting processes

- For simple cases,  $P(x|\mu)$  is known analytically and the confidence belt can be constructed analytically
  - Poisson counting process with a fixed background estimate,
  - Example: for  $P(x|s+b)$  with  $b=3.0$  known exactly

Confidence belt from  
68% and 90% central intervals

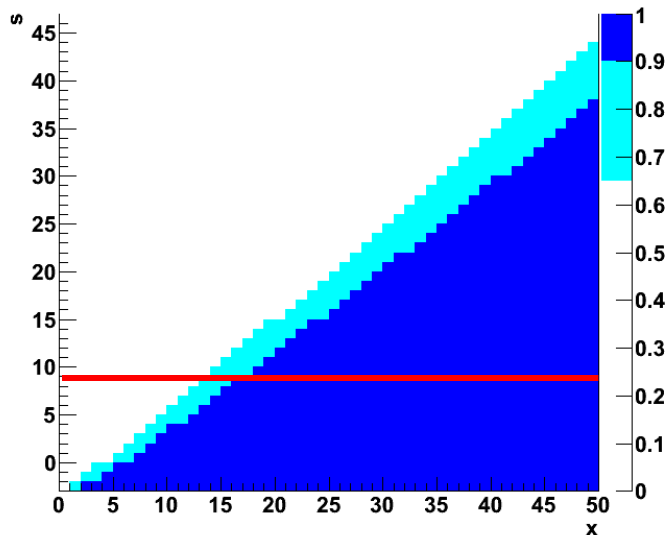


Confidence belt from  
68% and 90% upper limit



# Connection with hypothesis testing example

- Construction of confidence intervals and hypothesis testing closely connected.
- Going back to opening example:  
worked with  $P(x|\mu)$  with  $\mu=3$  to calculate p-value  $\rightarrow$   
Slice at  $\mu=3$  of confidence belt



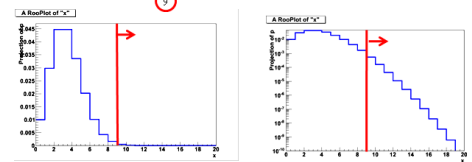
## Frequentist P – working out example #2

- Work out attempt #2 (aim to disprove Standard Model)
  - ‘Signal’ = SUSY ‘Background’ = Standard model

$$\begin{array}{ll} \text{Prediction} & N=3 \\ \text{Measurement} & N=9 \end{array} \quad \rightarrow \quad \begin{array}{ll} N(\text{bkg}) & = 3 \\ N(\text{sig+bkg}) & = 9 \end{array}$$

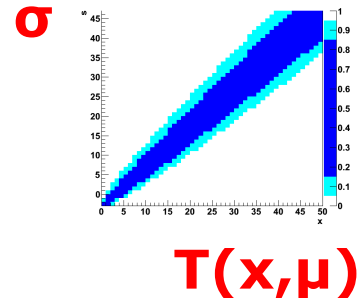
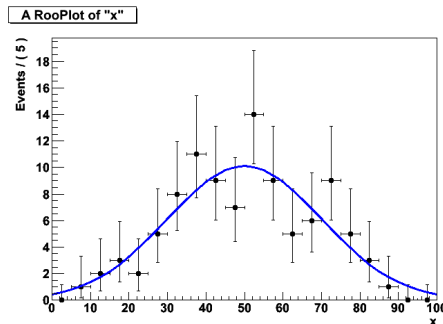
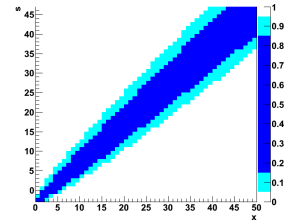
- Can we calculate probability that SM mimics SM+SUSY (i.e. result is a ‘false positive’)?
  - Calculation details depend on how measurement was done (fit, counting etc..)
  - Simplest case: counting experiment, Poisson process

$$p = \int_0^{\infty} \text{Poisson}(n, \mu=3) dn = 0.0038 \quad \text{= 'p value'}$$



# Confidence belts for non-counting data

- Confidence for simple counting experiment easy
  - Data = Single observable 'N',
  - Hypothesis: Poisson model  $P(N|\mathbf{s}+b)$  with  $b=\text{fixed}$
- What if a single measurement is a histogram?
  - Data = Histogram in 'x'
  - Hypothesis = Gaussian model  $G(x|\mu,\sigma)$  with  $\mu=\text{fixed}$
  - Parameter  $\sigma$  goes on 'y axis', what goes on 'x axis' of Neyman?



- Solution: you construct a test statistic  $T(x,\mu)$



# Confidence belts for non-trivial data

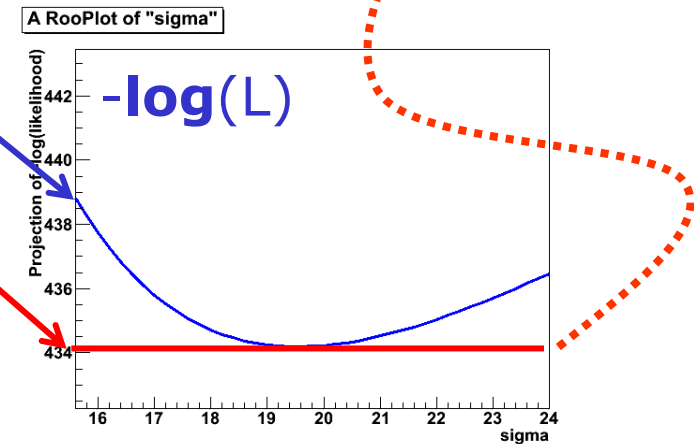
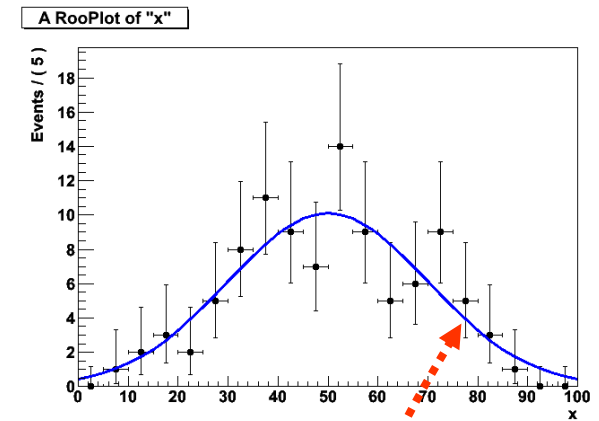
- Common choice of test statistic is a **Likelihood Ratio**
  - $\text{pdf}(x, \mu) = \text{Gaussian}(x, 50, \mu)$

Likelihood of data for model  
for a *given* value of  $\mu=1000$

$$LR(\vec{x}_{data}, \mu) = \frac{L(\vec{x}_{data}, \mu)}{L(\vec{x}_{data}, \hat{\mu})}$$

$L = \prod_{data} F(\vec{x}_i, \mu)$

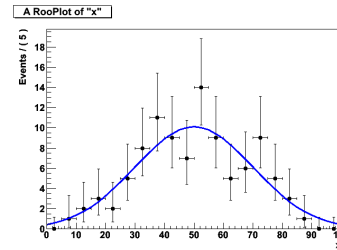
Likelihood of data for model  
at fitted value of  $\mu$



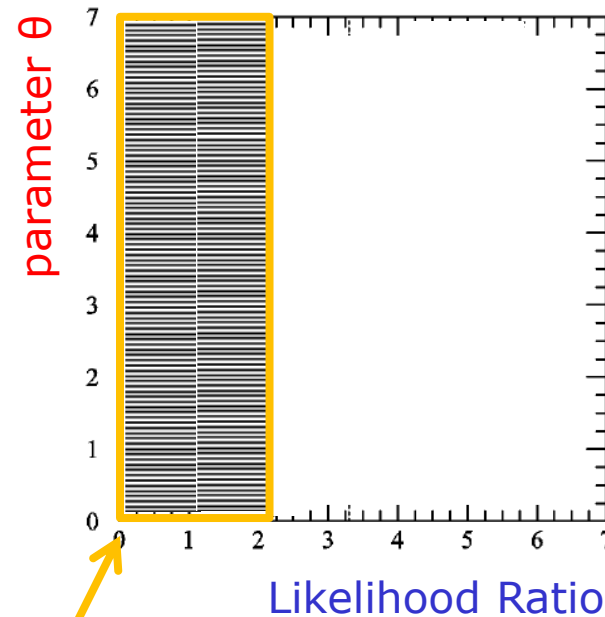
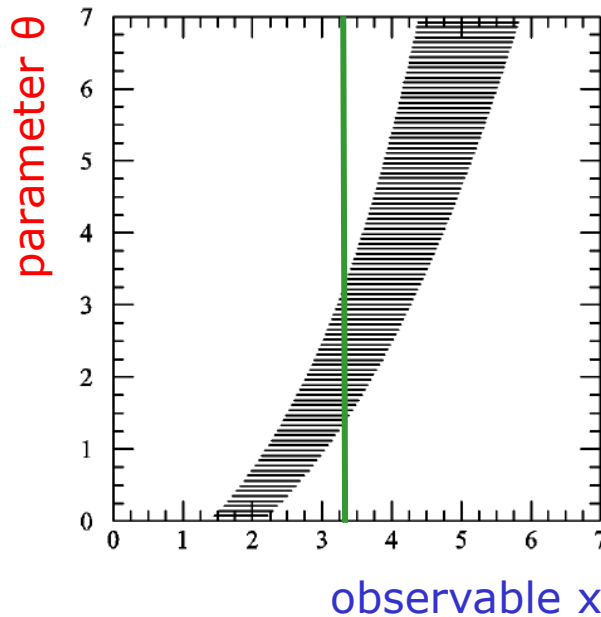
# Confidence belts for non-trivial data

- What will the confidence belt look like when replacing  $x \rightarrow LR(\vec{x}, \theta)$

$x=3.2$



$LR(x, \theta)$

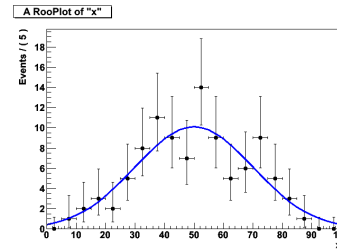


Confidence interval now range in LR

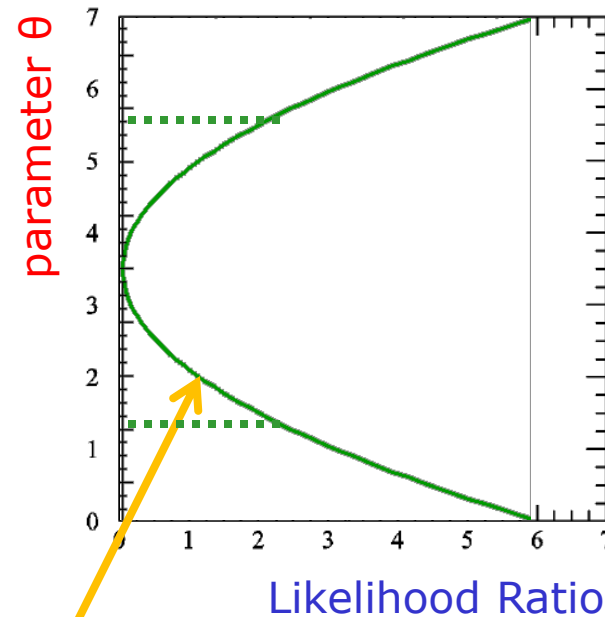
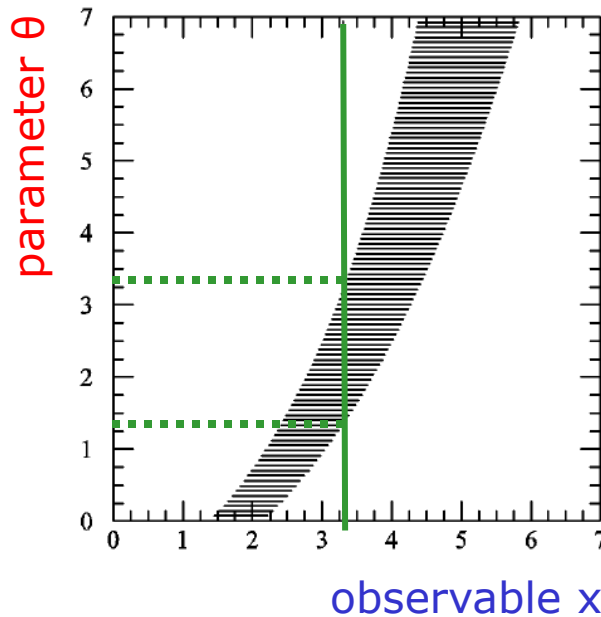
# Confidence belts for non-trivial data

- What will the confidence belt look like when replacing  $x \rightarrow LR(\vec{x}, \theta)$

$x=3.2$



$LR(x, \theta)$



**Measurement =  $LR(x_{\text{obs}}, \theta)$   
is now a function of  $\theta$**

# Confidence belts with Likelihood Ratio ordering rule

- Note that a confidence interval with a **Likelihood Ratio ordering rule** (i.e. acceptance interval is defined by a range in the LR) is exactly the **Feldman-Cousins** interval

Unified approach to the classical statistical analysis of small signals

Gary J. Feldman<sup>\*</sup>

*Department of Physics, Harvard University, Cambridge, Massachusetts 02138*

Robert D. Cousins<sup>†</sup>

*Department of Physics and Astronomy, University of California, Los Angeles, California 90095*

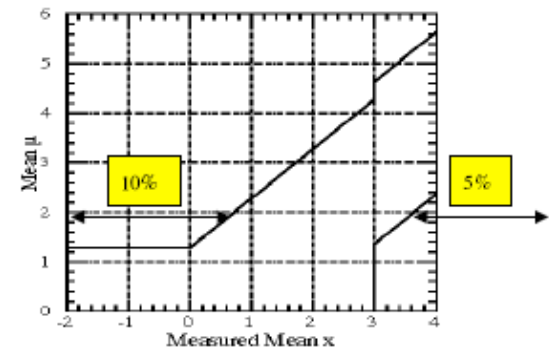
- One of the important features of FC that it provides a *unified method* for upper limits and central confidence intervals with good coverage
  - Upper limit at low  $x$ , central interval at higher
  - When choosing 'ad hoc' criteria to switch, good chance that your procedure doesn't have good coverage

## Flip-flopping

How might a typical physicist use these plots?

- (1) "If the result  $x < 3\sigma$ , I will quote an upper limit."
- (2) "If the result  $x > 3\sigma$ , I will quote central confidence interval."
- (3) "If the result  $x < 0$ , I will pretend I measured zero."

This results in the following:



In the range  $1.36 \leq \mu \leq 4.28$ , there is only 85% coverage!

Due to flip-flopping (deciding whether to use an upper limit or a central confidence region based on the data) **these are not valid confidence intervals.**

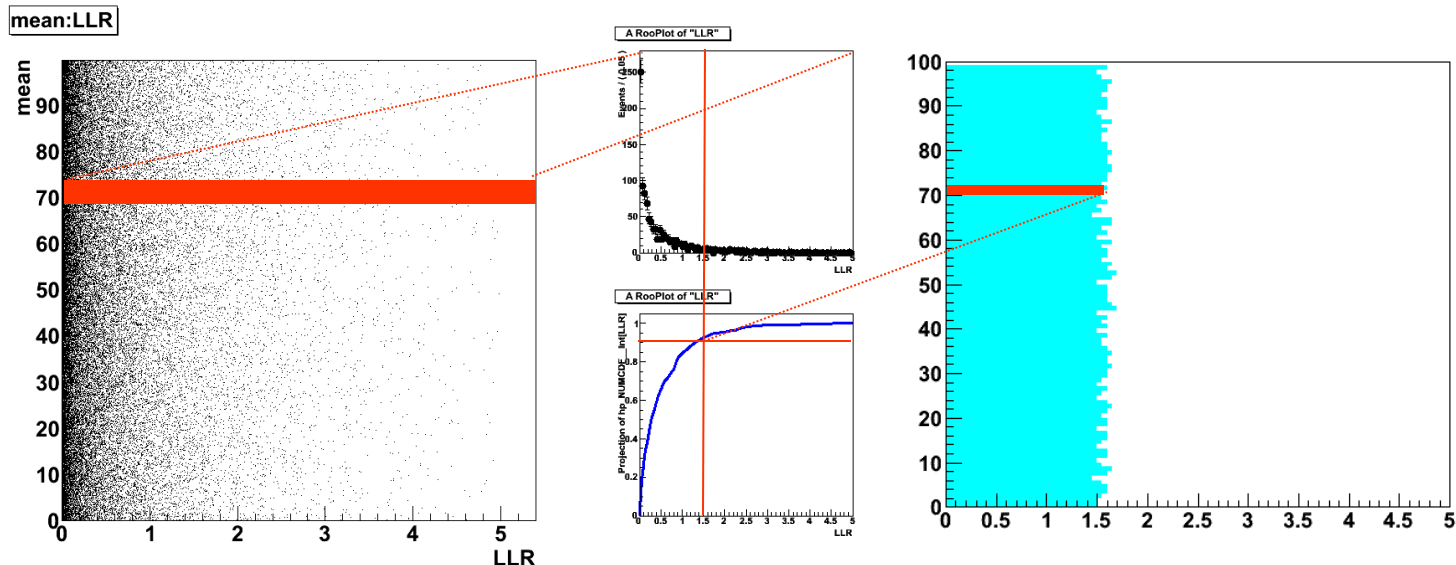


# Confidence belts with Likelihood Ratio ordering rule

- How can we determine the shape of the confidence belt in  $(LR, \mu)$  for random problem
  - In the case of the  $\text{Poisson}(x|s+b)$  confidence belt in  $(x, s)$  we could construct the belt directly from the p.d.f.
  - In rare cases you can do the same for a belt in  $(LR, s)$

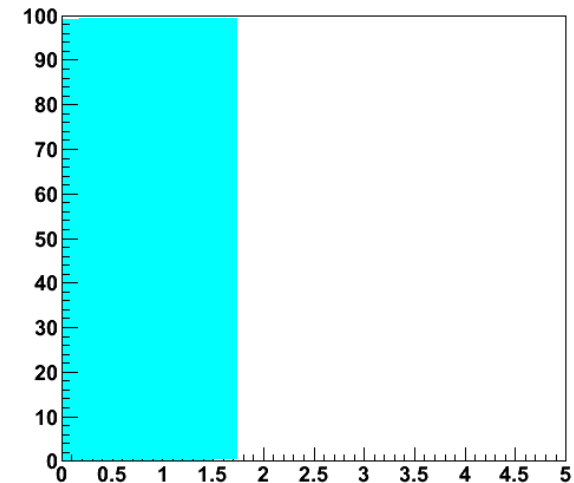
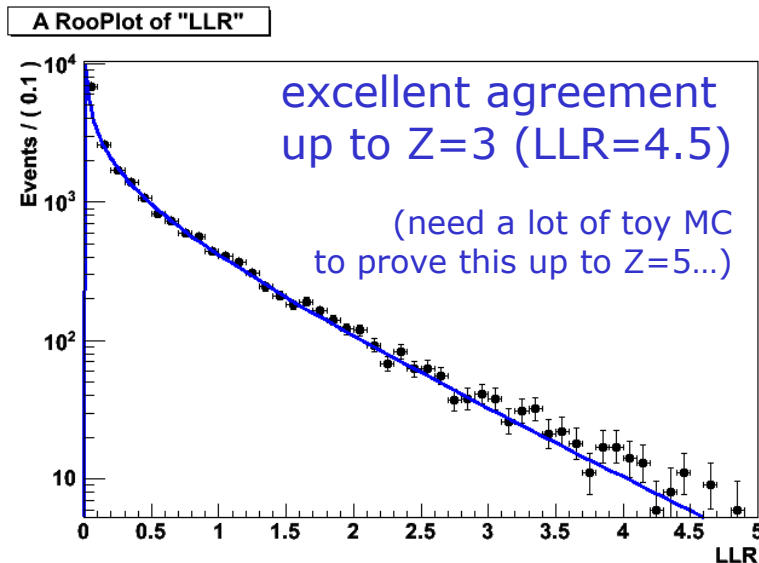
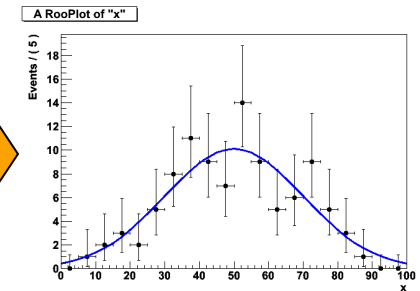
## 1. Calculation with toy-MC sampling

- For each  $\mu$  generate  $N$  samples of 'toy' data generated from the model  $F(x|\mu)$ . Calculate LR for each toy and construct distribution



# Confidence belts with Likelihood Ratio ordering rule

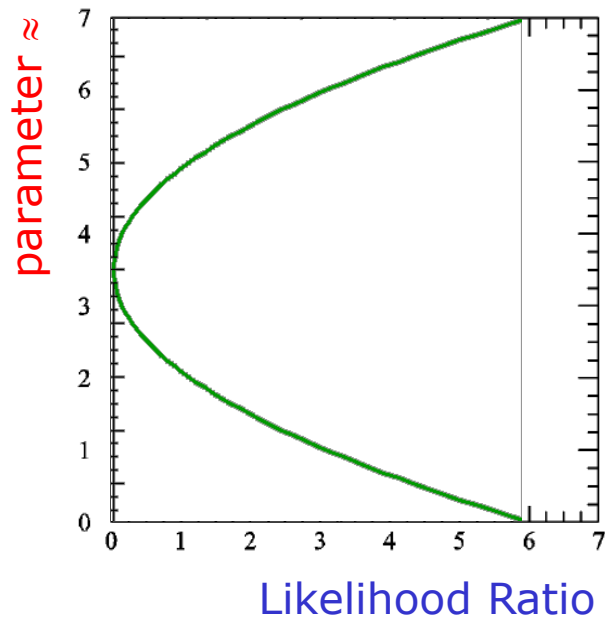
- Use asymptotic distribution of LR
  - Wilks theorem  $\rightarrow$  Asymptotic distribution of  $-\log(\text{LR})$  is chi-squared distribution  $\chi^2(2 \cdot \text{LLR}, n)$ , with  $n$  the number of parameters of interest ( $n=1$  in example shown)
  - Does **not** assume p.d.f.s are Gaussian
  - Example:  
LLR distribution from 100 event,  
20-bin measurement with Gaussian model  
from toy MC (histogram) vs asymptotic p.d.f



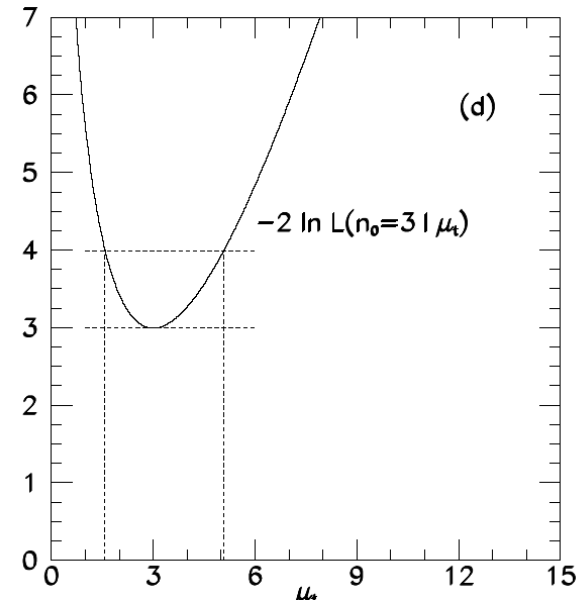
# Connection with likelihood ratio intervals

- If you assume the asymptotic distribution for LLR,
  - Then the confidence belt is exactly a box
  - And the constructed confidence interval can be simplified to finding the range in  $\mu$  where  $LLR = \frac{1}{2} \cdot Z^2$ 
    - This is exactly the MINOS error

FC interval with Wilks Theorem



MINOS / Likelihood ratio interval



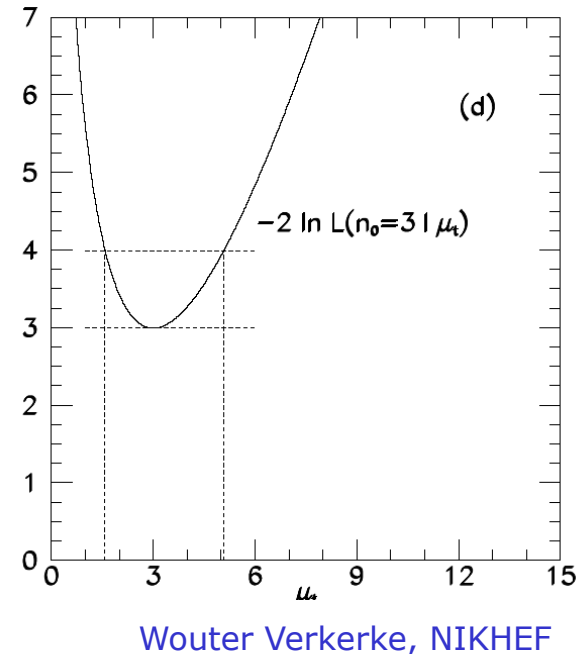
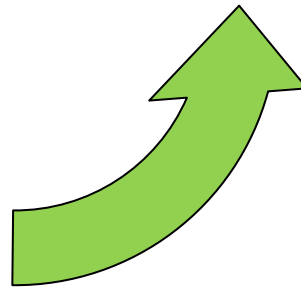
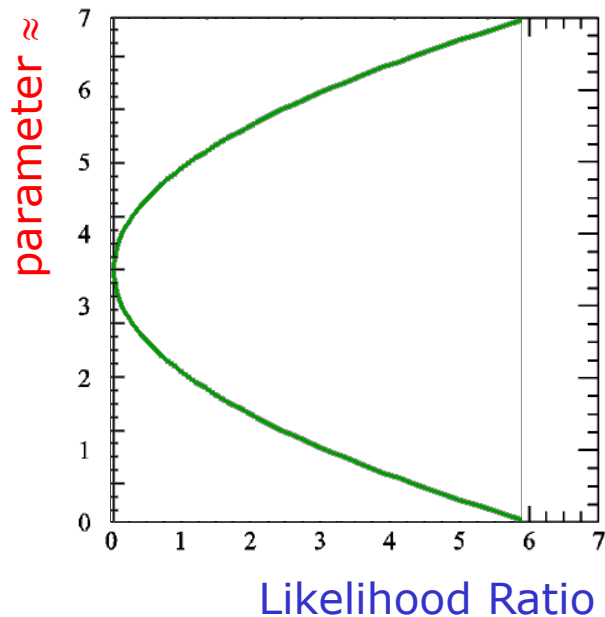
Wouter Verkerke, NIKHEF

# Likelihood (Ratio) Intervals

- Thus, after using maximum-likelihood method to obtain estimate  $\hat{u}$  which maximizes  $L(u)$ , one can obtain a likelihood interval  $[u_1, u_2]$  as the union of all  $u$  for which

$$LR(u) \leq Z^2, \text{ for } Z \text{ real.}$$

- But! Regularity conditions, in particular requirement that  $\hat{u}$  not be on the boundary, need to be carefully checked. (E.g., if  $u \geq 0$  on physical grounds, then  $\hat{u}=0$  requires care.)





## Likelihood-Ratio Interval example

- 68% C.L. likelihood-ratio interval for Poisson process with  $n=3$  observed:
- $L(\mu) = \mu^3 \exp(-\mu)/3!$
- Maximum at  $\mu = 3$ .
- $\Delta 2 \ln(L) = 1^2$  yields interval  $[1.58, 5.08]$

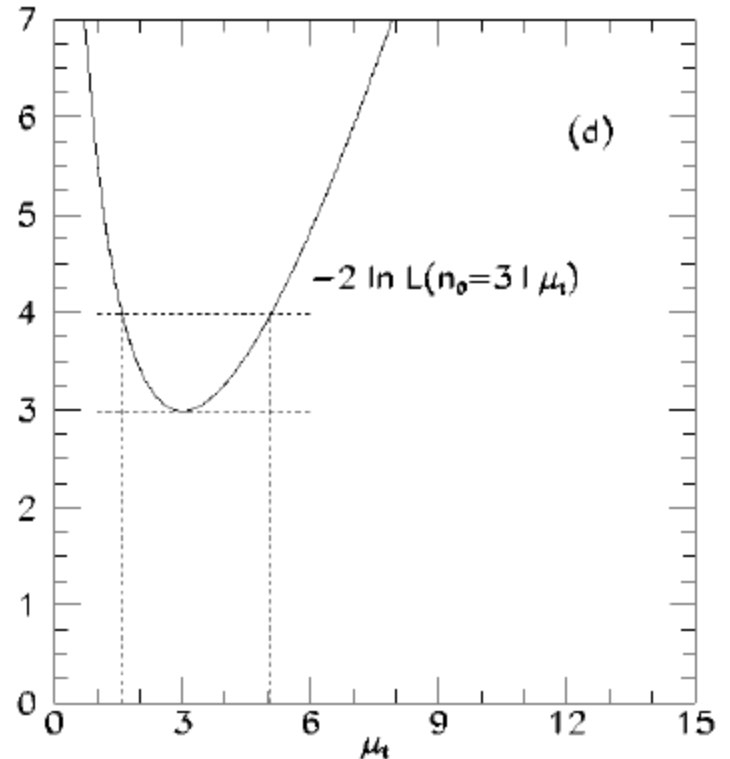
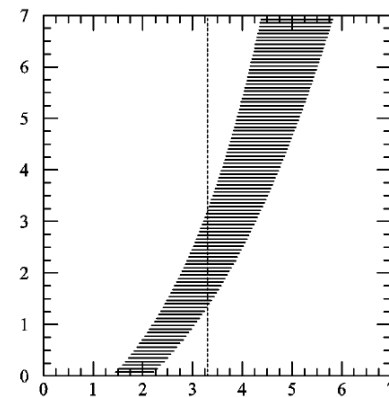
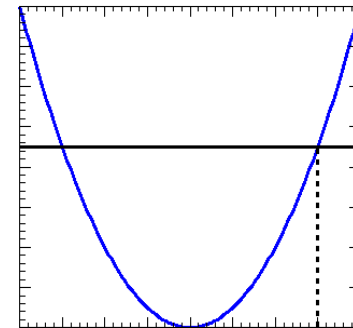
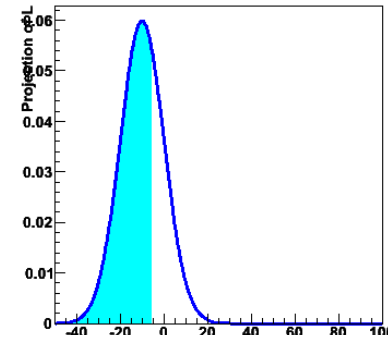


Figure from R. Cousins,  
Am. J. Phys. 63 398 (1995)

# U.L. in Poisson Process, $n=3$ observed: 3 ways

- **Bayesian interval**  
at 90% credibility:  
find  $\mu_u$  such that posterior  
probability  $p(\mu > \mu_u) = 0.1$ .
- **Likelihood ratio method** for  
approximate 90% C.L. U.L.:  
find  $\mu_u$  such that  $L(\mu_u) / L(3)$   
has prescribed value.
- **Frequentist one-sided 90%  
C.L. upper limit**: find  $\mu_u$  such  
that  $P(n \leq 3 \mid \mu_u) = 0.1$ .



## 68% intervals by various methods for Poisson process with $n=3$ observed

Method	Prior	Interval	Length	Coverage?
rms deviation $n \pm \sqrt{n}$	–	(1.27, 4.73)	3.46	no
Bayesian central	1	(2.09, <b>5.92</b> )	3.83	no
Bayesian shortest	1	(1.55, 5.15)	3.60	no
Bayesian central	$1/\mu$	( <b>1.37</b> , 4.64)	3.27	no
Bayesian shortest	$1/\mu$	(0.86, 3.85)	2.99	no
Likelihood ratio	–	(1.58, 5.08)	3.50	no
Frequentist central	–	( <b>1.37</b> , <b>5.92</b> )	4.55	yes
Frequentist shortest	–	(1.29, 5.25)	3.96	yes
Frequentist LR ordering	–	(1.10, 5.30)	4.20	yes

- NB: Frequentist intervals over-cover due to discreteness of  $n$  in this example
- Note that issues, divergences in outcome are usually more dramatic and important at high  $Z$  (e.g.  $5\sigma$  = 'discovery')

# Summary

- Three classes of inference (for limits and intervals)
  - **Bayesian** → Results in probability density function on true value. Prior knowledge always implicitly or explicitly assumed
  - **Frequentist** → Statement on frequency of obtained result (X% of time true value will be in interval)
  - **Likelihood** → Asymptotically identical to Frequentist interval with LR ordering rule (Feldman Cousins, Wilks Theorem)
- For 'simple problems' (high statistics, limits at  $\ll 5\sigma$ ) all procedures usually give comparable answers
- For 'difficult problems' (low stats, high limits) answer will diverge
  - See Poisson  $n=3$  for low statistics example
  - Results depends on precise definition of question asked, which is different for each described technique