

The non-autonomous chiral model and the Ernst equation of general relativity in the bidifferential calculus framework

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Introduction

- ▶ Classical *integrable systems* are non-linear equations, which allow the construction of large families of exact solutions
 - ▶ For certain equations these include *solitons*:
Localized waves, which keep their shape after interactions

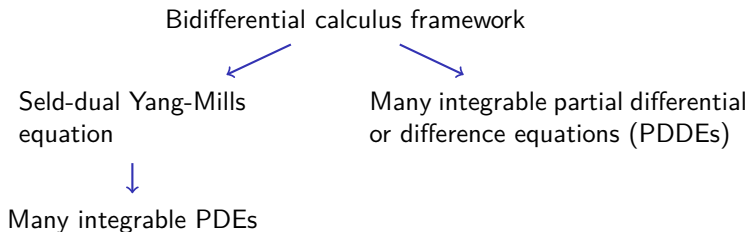
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- ▶ *Stationary axially symmetric (electro-)vacuum Einstein* equations are “integrable” [Belinski and Zakharov 1978; Maison 1979]
 - ▶ Rotating (charged) black holes can be understood as “solitons”
 - ▶ Many methods to construct solutions known

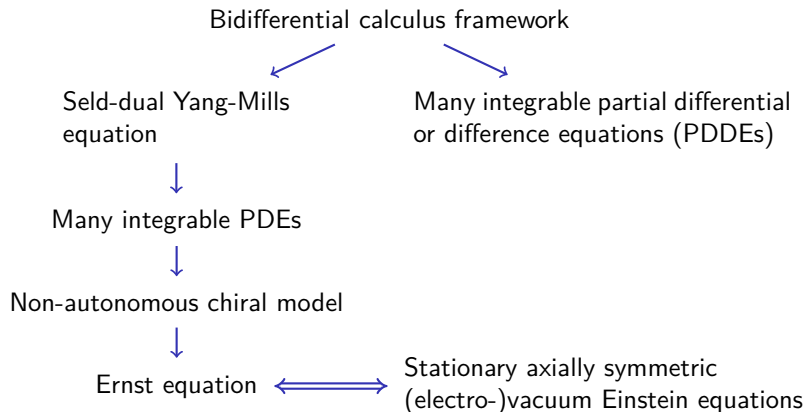
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 - ▶ Many methods to construct solutions known
- ▶ *Bidifferential calculus* framework is an abstract characterization of integrable systems [Dimakis and Müller-Hoissen 2000, 2009]
 - ▶ Solution generating methods can be formulated independent of examples

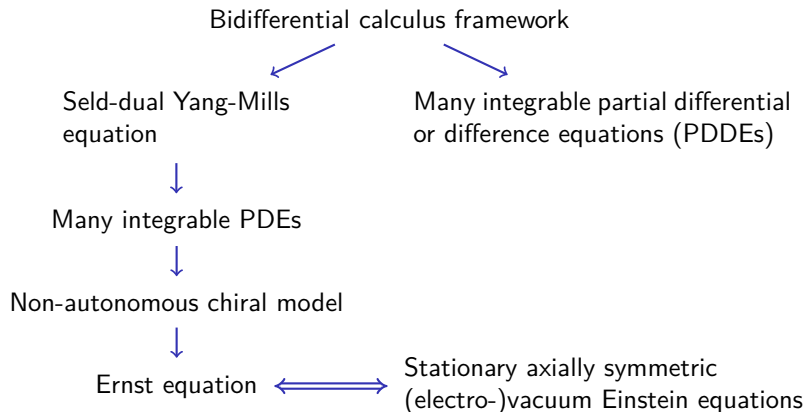
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- ▶ Outline of the talk:
 - ▶ Powerful non-iterative solution generating method in the bidifferential calculus framework
 - ▶ Application to non-autonomous chiral model and Ernst equation

Basic definitions

- ▶ A *bidifferential calculus* (Ω, d, \bar{d}) is given by:
 - ▶ A unital associative graded \mathbb{C} -algebra $\Omega = \bigoplus_{r \geq 0} \Omega^r$
 - ▶ Two graded derivations of degree one $d, \bar{d} : \Omega^r \rightarrow \Omega^{r+1}$ satisfying

$$d_\kappa^2 = 0 \quad \forall \kappa \in \mathbb{C},$$

where $d_\kappa := \bar{d} - \kappa d$, and the graded Leibniz rule

$$d_\kappa(\chi\chi') = (d_\kappa \chi)\chi' + (-1)^r \chi d_\kappa \chi' \quad \forall \kappa \in \mathbb{C}$$

for $\chi \in \Omega^r, \chi' \in \Omega$

- ▶ Generalization of differential forms on a manifold
 - ▶ Keep nice properties of exterior derivative

Bidifferential calculus formulation of “integrable” PDDE

- ▶ Choice of:
 - ▶ Bidifferential calculus (Ω, d, \bar{d})
 - ▶ Parameterization of a 1-form $\mathbb{A} \in \Omega^1$ by variables of PDDE

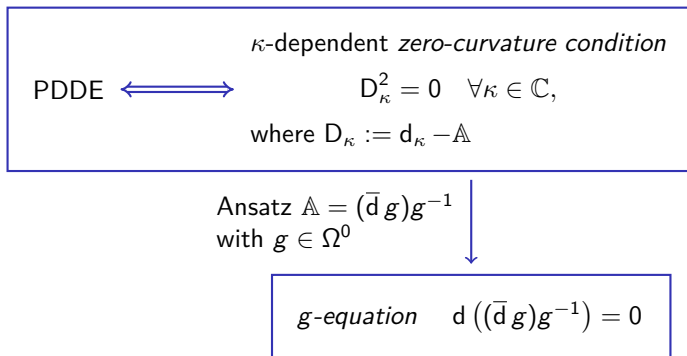
Such that:

	<i>κ-dependent zero-curvature condition</i>
PDDE	\longleftrightarrow
	$D_{\kappa}^2 = 0 \quad \forall \kappa \in \mathbb{C},$
	where $D_{\kappa} := d_{\kappa} - \mathbb{A}$

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Such that:



- ▶ Other ansätze for \mathbb{A} possible, but not considered in this talk

Solution generating method

- ▶ $\Omega = \Omega^0 \otimes \wedge(\mathbb{C}^N)$ with Ω^0 all matrices over some unital algebra \mathcal{B}
- ▶ Theorem:

$\mathbf{P}, \mathbf{R}, \mathbf{X} \in \mathcal{B}^{n \times n}$ invertible solutions of

$$\bar{d}\mathbf{P} = (d\mathbf{P})\mathbf{P}, \quad \bar{d}\mathbf{R} = \mathbf{R}(d\mathbf{R}), \quad \bar{d}\mathbf{X} = (d\mathbf{X})\mathbf{P} - (d\mathbf{R})\mathbf{X},$$

$$\mathbf{X}\mathbf{P} - \mathbf{R}\mathbf{X} = \mathbf{V}\mathbf{U}$$

with d- and \bar{d} -constant $\mathbf{U} \in \mathcal{B}^{m \times n}$, $\mathbf{V} \in \mathcal{B}^{n \times m}$

$\Rightarrow g = I_m + \mathbf{U}(\mathbf{R}\mathbf{X})^{-1}\mathbf{V} \in \mathcal{B}^{m \times m}$ solves g -equation

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- ▶ Powerful solution generating method
 - ▶ Typically contains multi-soliton solutions
 - ▶ Solutions parametrized by matrix data $(\mathbf{P}, \mathbf{R}, \mathbf{U}, \mathbf{V})$
 - ▶ Non-linear superposition corresponds to block-wise composition of matrix data
- ▶ Special case with d- and \bar{d} -constant \mathbf{P}, \mathbf{R} well known [Dimakis and Müller-Hoissen 2009, 2010; Dimakis, NK and Müller-Hoissen 2011]

Proof of the solution generating method

- ▶ Proof by short elementary computation
- ▶ Alternative proof:

$$\begin{array}{ccc}
 \mathcal{B}^{n \times n} & \text{Darboux} & \mathcal{B}^{n \times n} \\
 \Downarrow & \text{transformation} & \Downarrow \\
 \mathbf{g} = \mathbf{R}^{-1} & \xrightarrow{\text{"adds soliton"}} & \mathbf{g}' = \mathbf{X} \mathbf{P} \mathbf{X}^{-1} \mathbf{R}^{-1} \\
 & & \downarrow \\
 & & \text{Projection} \\
 & & \text{[Marchenko 1988]} \\
 & & \downarrow \\
 & & \mathbf{g} = (\mathbf{U} \mathbf{g}'^{-1} \mathbf{V})^{-1} \\
 & & \cap \\
 & & \mathcal{B}^{m \times m}
 \end{array}$$

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Projection
[Marchenko 1988]

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 \Downarrow \\
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- ▶ Interpretation:

Typically
"1- or 2-soliton solution"
of g -equation in $\mathcal{B}^{n \times n}$

$$n > m$$

"Multi-soliton solution"
of g -equation in $\mathcal{B}^{m \times m}$

Bidifferential calculus for non-autonomous chiral model

- ▶ Finding a bidifferential calculus formulation for a given “integrable” PDDE is a difficult problem (like finding a Lax pair)
- ▶ Choose bidifferential calculus (Ω, d, \bar{d}) :
 - ▶ $\Omega = \Omega^0 \otimes \wedge(\mathbb{C}^2)$ with $\Omega^0 = \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{C})^{m \times m}$
 - ▶ For $f \in \Omega^0$ define

$$df = -f_z \zeta_1 + e^\theta (f_\rho - \rho^{-1} f_\theta) \zeta_2,$$

$$\bar{d}f = e^{-\theta} (f_\rho + \rho^{-1} f_\theta) \zeta_1 + f_z \zeta_2$$

with a basis ζ_1, ζ_2 of $\wedge^1(\mathbb{C}^2)$

Parameterization of $g \in \Omega^0$:

- ▶ $g = e^{c\theta} \tilde{g}(z, \rho)$ with $c \in \mathbb{C}$

g-equation

$$d((\bar{d}g)g^{-1}) = 0$$



Non-autonomous chiral model

$$(\rho \tilde{g}_z \tilde{g}^{-1})_z + (\rho \tilde{g}_\rho \tilde{g}^{-1})_\rho = 0$$

for $m \times m$ matrix \tilde{g}

Application of the solution generating method

- ▶ Equations for \mathbf{P}, \mathbf{R} essentially reduce to $n \times n$ matrix equation

$$\tilde{\mathbf{P}}^2 - 2\rho^{-1}(z\mathbf{I}_n + \mathbf{B})\tilde{\mathbf{P}} - \mathbf{I}_n = 0$$

for $\tilde{\mathbf{P}}$ with a constant matrix \mathbf{B} (respectively for $\tilde{\mathbf{R}}$ with \mathbf{B}')

- ▶ If $\text{spec } \tilde{\mathbf{P}} \cap \text{spec } \tilde{\mathbf{R}} = \emptyset$, it only remains to solve a Sylvester equation

$$\tilde{\mathbf{X}}\tilde{\mathbf{P}} - \tilde{\mathbf{R}}\tilde{\mathbf{X}} = \mathbf{V}\mathbf{U}$$

for the $n \times n$ matrix $\tilde{\mathbf{X}}$ (a unique solution exists)

$\Rightarrow \tilde{\mathbf{g}} = \mathbf{I}_m + \mathbf{U}(\tilde{\mathbf{R}}\tilde{\mathbf{X}})^{-1}\mathbf{V}$ solves non-autonomous chiral model

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- ▶ Example:

$\tilde{\mathbf{P}} = (p_i\delta_{ij}), \tilde{\mathbf{R}} = (r_i\delta_{ij})$ diagonal:

$$p_i = \rho^{-1} \left(z + b_i + j_i \sqrt{(z + b_i)^2 + \rho^2} \right), \quad \tilde{\mathbf{X}}_{ij} = \frac{(\mathbf{V}\mathbf{U})_{ij}}{p_j - r_i}$$

with constants b_i and $j_i \in \{\pm 1\}$ (respectively for r_i with b'_i, j'_i)
such that $\{p_i\} \cap \{r_i\} = \emptyset$

Connection with general relativity

Stationary axially symmetric vacuum Einstein equations

↕ [Ernst 1968]

Ernst equation for complex scalar function \mathcal{E}

$$(\operatorname{Re} \mathcal{E})(\mathcal{E}_{\rho\rho} + \rho^{-1}\mathcal{E}_\rho + \mathcal{E}_{zz}) = (\mathcal{E}_\rho)^2 + (\mathcal{E}_z)^2$$

↕ [Witten 1979]

Non-autonomous chiral model with $m = 2$ and

$$\tilde{g} = \frac{2}{\mathcal{E} + \bar{\mathcal{E}}} \begin{pmatrix} 1 & \frac{i}{2}(\mathcal{E} - \bar{\mathcal{E}}) \\ \frac{i}{2}(\mathcal{E} - \bar{\mathcal{E}}) & \mathcal{E}\bar{\mathcal{E}} \end{pmatrix}$$

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- ▶ Parameterization of \tilde{g} equivalent to reduction conditions

$$\tilde{g}^\dagger = \tilde{g}, \quad (\gamma\tilde{g})^2 = I_m, \quad \operatorname{tr}(\gamma\tilde{g}) = m - 2$$

and $\gamma^\dagger = \gamma$, $\gamma^2 = I_m$ with constant $m \times m$ matrix γ [Gürses 1984]

- ▶ $m = 3$: Stationary axially symmetric electro-vacuum space-times

Kerr-NUT solution

- ▶ Implement reduction conditions on solutions in example above
- ▶ $m = n = 2$:

Explicit analysis of solution \tilde{g} implies

$$\tilde{\mathbf{P}} = \begin{pmatrix} p(b, j) & 0 \\ 0 & p(b, -j) \end{pmatrix}, \quad \tilde{\mathbf{R}} = \begin{pmatrix} r(b', j') & 0 \\ 0 & r(b', -j') \end{pmatrix},$$

$$\mathbf{U} = \begin{pmatrix} 1 & -u \\ u & 1 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} 1 & v \\ -v & 1 \end{pmatrix}$$

with one of the following conditions on the parameters:

- ▶ b, b', u, v real:
Non-extreme Kerr-NUT space-time
 (includes rotating black hole)
- ▶ $b = \bar{b}', j = -j', v = \bar{u}$:
Hyperextreme Kerr-NUT space-time
 (naked singularity)

Multi-Kerr-NUT solutions

- ▶ $m = 2, n = 2N$:

Superpose N non-extreme or N hyperextreme Kerr-NUT solutions by composing their matrix data

$$\tilde{\mathbf{P}}_i = \begin{pmatrix} p_i(b_i, j_i) & 0 \\ 0 & p_i(b_i, -j_i) \end{pmatrix}, \quad \tilde{\mathbf{R}}_i = \begin{pmatrix} r_i(b'_i, j'_i) & 0 \\ 0 & r_i(b'_i, -j'_i) \end{pmatrix},$$

$$\mathbf{U}_i = \begin{pmatrix} 1 & -u_i \\ u_i & 1 \end{pmatrix}, \quad \mathbf{V}_i = \begin{pmatrix} 1 & v_i \\ -v_i & 1 \end{pmatrix}$$

with $p_i \neq r_k$ to

$$\tilde{\mathbf{P}} = \text{block-diag}(\tilde{\mathbf{P}}_1, \dots, \tilde{\mathbf{P}}_N), \quad \mathbf{U} = (\mathbf{U}_1 \quad \dots \quad \mathbf{U}_N), \quad \mathbf{V} = \begin{pmatrix} \mathbf{V}_1 \\ \vdots \\ \mathbf{V}_N \end{pmatrix}$$

$$\tilde{\mathbf{R}} = \text{block-diag}(\tilde{\mathbf{R}}_1, \dots, \tilde{\mathbf{R}}_N),$$

- ▶ Reduction conditions on \tilde{g} ensured by conditions on matrix data $(\tilde{\mathbf{P}}, \tilde{\mathbf{R}}, \mathbf{U}, \mathbf{V})$, these conditions are fulfilled because they hold for each block $(\tilde{\mathbf{P}}_i, \tilde{\mathbf{R}}_i, \mathbf{U}_i, \mathbf{V}_i)$
- ▶ $m = 3$: Hyperextreme multi-Demiański-Newman solution (charged generalization of multi-Kerr-NUT) obtained analogously

Conclusions

- ▶ Generalization of solution generating method to non-constant \mathbf{P}, \mathbf{R}
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 - ▶ Solutions with non-diagonal $\tilde{\mathbf{P}}, \tilde{\mathbf{R}}$ known (some are limits of solutions with diagonal $\tilde{\mathbf{P}}, \tilde{\mathbf{R}}$, no systematic exploration yet)
 - ▶ Also solutions with $\text{spec } \tilde{\mathbf{P}} \cap \text{spec } \tilde{\mathbf{R}} \neq \emptyset$ possible?

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- ▶ Non-autonomous chiral model and Ernst equation addressed in new way using bidifferential calculus
 - ▶ Established solid foundations, but still much to explore!