The non-autonomous chiral model and the Ernst equation of general relativity in the bidifferential calculus framework

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Introduction

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  - For certain equations these include *solitons*:
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  - Rotating (charged) black holes can be understood as "solitons"
  - Many methods to construct solutions known
- *Bidifferential calculus* framework is an abstract characterization of integrable systems [Dimakis and Müller-Hoissen 2000, 2009]
  - Solution generating methods can be formulated independent of examples

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Bidifferential calculus framework

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Seld-dual Yang-Mills equation

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Many integrable PDEs

Many integrable partial differential or difference equations (PDDEs)

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- Outline of the talk:
  - Powerful non-iterative solution generating method in the bidifferential calculus framework
  - Application to non-autonomous chiral model and Ernst equation

Basic definitions

- A *bidifferential calculus* $(\Omega, \mathrm{d}, \bar{\mathrm{d}})$ is given by:
  - A unital associative graded $\mathbb{C}$-algebra $\Omega = \bigoplus_{r \geq 0} \Omega^r$
  - Two graded derivations of degree one $\mathrm{d}, \bar{\mathrm{d}} : \Omega^r \to \Omega^{r+1}$ satisfying

    $$\mathrm{d}_\kappa^2 = 0 \quad \forall \kappa \in \mathbb{C},$$

    where $\mathrm{d}_\kappa := \bar{\mathrm{d}} - \kappa \, \mathrm{d}$, and the graded Leibniz rule

    $$\mathrm{d}_\kappa(\chi \chi') = (\mathrm{d}_\kappa \, \chi)\chi' + (-1)^r \chi \, \mathrm{d}_\kappa \, \chi' \quad \forall \kappa \in \mathbb{C}$$

    for $\chi \in \Omega^r$, $\chi' \in \Omega$
- Generalization of differential forms on a manifold
  - Keep nice properties of exterior derivative

Bidifferential calculus formulation of “integrable” PDDE

- Choice of:
  - Bidifferential calculus $(\Omega, \mathrm{d}, \bar{\mathrm{d}})$
  - Parameterization of a 1-form $\mathbb{A} \in \Omega^1$ by variables of PDDE

Such that:

PDDE $\Longleftrightarrow$ κ-dependent *zero-curvature condition*

$$\mathrm{D}_\kappa^2 = 0 \quad \forall \kappa \in \mathbb{C},$$

where $\mathrm{D}_\kappa := \mathrm{d}_\kappa - \mathbb{A}$

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↓ Ansatz $\mathbb{A} = (\bar{\mathrm{d}}\, g)g^{-1}$ with $g \in \Omega^0$

g-equation $\quad \mathrm{d}\left((\bar{\mathrm{d}}\, g)g^{-1}\right) = 0$

- Other ansätze for $\mathbb{A}$ possible, but not considered in this talk

Solution generating method

- $\Omega = \Omega^0 \otimes \bigwedge(\mathbb{C}^N)$ with Ω^0 all matrices over some unital algebra $\mathcal{B}$
- Theorem:

$\boldsymbol{P}, \boldsymbol{R}, \boldsymbol{X} \in \mathcal{B}^{n\times n}$ invertible solutions of

$$\bar{\mathrm{d}}\,\boldsymbol{P} = (\mathrm{d}\,\boldsymbol{P})\boldsymbol{P}, \quad \bar{\mathrm{d}}\,\boldsymbol{R} = \boldsymbol{R}(\mathrm{d}\,\boldsymbol{R}), \quad \bar{\mathrm{d}}\,\boldsymbol{X} = (\mathrm{d}\,\boldsymbol{X})\boldsymbol{P} - (\mathrm{d}\,\boldsymbol{R})\boldsymbol{X},$$
$$\boldsymbol{X}\boldsymbol{P} - \boldsymbol{R}\boldsymbol{X} = \boldsymbol{V}\boldsymbol{U}$$

with d- and $\bar{\mathrm{d}}$-constant $\boldsymbol{U} \in \mathcal{B}^{m\times n}$, $\boldsymbol{V} \in \mathcal{B}^{n\times m}$
$\Longrightarrow g = I_m + \boldsymbol{U}(\boldsymbol{R}\boldsymbol{X})^{-1}\boldsymbol{V} \in \mathcal{B}^{m\times m}$ solves g-equation

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- Powerful solution generating method
  - Typically contains multi-soliton solutions
  - Solutions parametrized by matrix data $(\boldsymbol{P}, \boldsymbol{R}, \boldsymbol{U}, \boldsymbol{V})$
  - Non-linear superposition corresponds to block-wise composition of matrix data
- Special case with d- and $\bar{\mathrm{d}}$-constant $\boldsymbol{P}, \boldsymbol{R}$ well known [Dimakis and Müller-Hoissen 2009, 2010; Dimakis, NK and Müller-Hoissen 2011]

Proof of the solution generating method

- Proof by short elementary computation
- Alternative proof:

$$\begin{array}{ccc} \mathcal{B}^{n\times n} & & \mathcal{B}^{n\times n} \\ \cup & & \cup \\ \boldsymbol{g} = \boldsymbol{R}^{-1} & \xrightarrow{\text{Darboux transformation “adds soliton”}} & \boldsymbol{g}' = \boldsymbol{X}\boldsymbol{P}\boldsymbol{X}^{-1}\boldsymbol{R}^{-1} \end{array}$$

Projection [Marchenko 1988] $\downarrow$

$$\begin{array}{c} g = (\boldsymbol{U}\boldsymbol{g}'^{-1}\boldsymbol{V})^{-1} \\ \cap \\ \mathcal{B}^{m\times m} \end{array}$$

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- Interpretation:

Typically “1- or 2-soliton solution” of g-equation in $\mathcal{B}^{n\times n}$

$\Big\downarrow$ $n > m$

“Multi-soliton solution” of g-equation in $\mathcal{B}^{m\times m}$

Bidifferential calculus for non-autonomous chiral model

- Finding a bidifferential calculus formulation for a given "integrable" PDDE is a difficult problem (like finding a Lax pair)
- Choose bidifferential calculus $(\Omega, \mathrm{d}, \bar{\mathrm{d}})$:
  - $\Omega = \Omega^0 \otimes \bigwedge(\mathbb{C}^2)$ with $\Omega^0 = \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{C})^{m\times m}$
  - For $f \in \Omega^0$ define

$$\mathrm{d}\, f = -f_z \zeta_1 + e^{\theta}(f_\rho - \rho^{-1} f_\theta)\zeta_2,$$
$$\bar{\mathrm{d}}\, f = e^{-\theta}(f_\rho + \rho^{-1} f_\theta)\zeta_1 + f_z \zeta_2$$

with a basis ζ_1, ζ_2 of $\bigwedge^1(\mathbb{C}^2)$

Parameterization of $g \in \Omega^0$:

- $g = e^{c\theta}\tilde{g}(z,\rho)$ with $c \in \mathbb{C}$

g-equation		*Non-autonomous chiral model*
$\mathrm{d}\left((\bar{\mathrm{d}}\, g)g^{-1}\right) = 0$	$\Longleftrightarrow$	$(\rho\tilde{g}_z\tilde{g}^{-1})_z + (\rho\tilde{g}_\rho\tilde{g}^{-1})_\rho = 0$ for $m \times m$ matrix $\tilde{g}$

Application of the solution generating method

- Equations for $\boldsymbol{P}, \boldsymbol{R}$ essentially reduce to $n \times n$ matrix equation

$$\tilde{\boldsymbol{P}}^2 - 2\rho^{-1}(z\boldsymbol{I}_n + \boldsymbol{B})\tilde{\boldsymbol{P}} - \boldsymbol{I}_n = 0$$

  for $\tilde{\boldsymbol{P}}$ with a constant matrix $\boldsymbol{B}$ (respectively for $\tilde{\boldsymbol{R}}$ with $\boldsymbol{B}'$)
- If $\operatorname{spec} \tilde{\boldsymbol{P}} \cap \operatorname{spec} \tilde{\boldsymbol{R}} = \emptyset$, it only remains to solve a Sylvester equation

$$\tilde{\boldsymbol{X}}\tilde{\boldsymbol{P}} - \tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}} = \boldsymbol{V}\boldsymbol{U}$$

  for the $n \times n$ matrix $\tilde{\boldsymbol{X}}$ (a unique solution exists)

$\Longrightarrow$ $\tilde{g} = I_m + \boldsymbol{U}(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}})^{-1}\boldsymbol{V}$ solves non-autonomous chiral model

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- Example:

$\tilde{\boldsymbol{P}} = (p_i\delta_{ij})$, $\tilde{\boldsymbol{R}} = (r_i\delta_{ij})$ diagonal:

$$p_i = \rho^{-1}\left(z + b_i + j_i\sqrt{(z+b_i)^2 + \rho^2}\right), \quad \tilde{\boldsymbol{X}}_{ij} = \frac{(\boldsymbol{V}\boldsymbol{U})_{ij}}{p_j - r_i}$$

with constants b_i and $j_i \in \{\pm 1\}$ (respectively for r_i with b_i', j_i') such that $\{p_i\} \cap \{r_i\} = \emptyset$

Connection with general relativity

Stationary axially symmetric vacuum Einstein equations

$\Updownarrow$ [Ernst 1968]

Ernst equation for complex scalar function $\mathcal{E}$

$$(\mathrm{Re}\,\mathcal{E})(\mathcal{E}_{\rho\rho} + \rho^{-1}\mathcal{E}_\rho + \mathcal{E}_{zz}) = (\mathcal{E}_\rho)^2 + (\mathcal{E}_z)^2$$

$\Updownarrow$ [Witten 1979]

Non-autonomous chiral model with $m = 2$ and

$$\tilde{g} = \frac{2}{\mathcal{E} + \overline{\mathcal{E}}} \begin{pmatrix} 1 & \frac{i}{2}(\mathcal{E} - \overline{\mathcal{E}}) \\ \frac{i}{2}(\mathcal{E} - \overline{\mathcal{E}}) & \mathcal{E}\overline{\mathcal{E}} \end{pmatrix}$$

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- Parameterization of $\tilde{g}$ equivalent to reduction conditions

$$\tilde{g}^\dagger = \tilde{g}, \quad (\gamma\tilde{g})^2 = I_m, \quad \mathrm{tr}\,(\gamma\tilde{g}) = m - 2$$

  and $\gamma^\dagger = \gamma$, $\gamma^2 = I_m$ with constant $m \times m$ matrix γ [Gürses 1984]
- $m = 3$: Stationary axially symmetric electro-vacuum space-times

Kerr-NUT solution

- Implement reduction conditions on solutions in example above
- $m = n = 2$:
  Explicit analysis of solution $\tilde{g}$ implies

$$\tilde{\boldsymbol{P}} = \begin{pmatrix} p(b,j) & 0 \\ 0 & p(b,-j) \end{pmatrix}, \quad \tilde{\boldsymbol{R}} = \begin{pmatrix} r(b',j') & 0 \\ 0 & r(b',-j') \end{pmatrix},$$

$$\boldsymbol{U} = \begin{pmatrix} 1 & -u \\ u & 1 \end{pmatrix}, \quad \boldsymbol{V} = \begin{pmatrix} 1 & v \\ -v & 1 \end{pmatrix}$$

  with one of the following conditions on the parameters:
  - b, b', u, v real:
    Non-extreme Kerr-NUT space-time
    (includes rotating black hole)
  - $b = \overline{b'}$, $j = -j'$, $v = \overline{u}$:
    Hyperextreme Kerr-NUT space-time
    (naked singularity)

Multi-Kerr-NUT solutions

- $m = 2$, $n = 2N$:
  Superpose N non-extreme or N hyperextreme Kerr-NUT solutions by composing their matrix data

$$\tilde{\boldsymbol{P}}_i = \begin{pmatrix} p_i(b_i, j_i) & 0 \\ 0 & p_i(b_i, -j_i) \end{pmatrix}, \quad \tilde{\boldsymbol{R}}_i = \begin{pmatrix} r_i(b'_i, j'_i) & 0 \\ 0 & r_i(b'_i, -j'_i) \end{pmatrix},$$

$$\boldsymbol{U}_i = \begin{pmatrix} 1 & -u_i \\ u_i & 1 \end{pmatrix}, \quad \boldsymbol{V}_i = \begin{pmatrix} 1 & v_i \\ -v_i & 1 \end{pmatrix}$$

  with $p_i \neq r_k$ to

$$\tilde{\boldsymbol{P}} = \text{block-diag}\,(\tilde{\boldsymbol{P}}_1, \ldots, \tilde{\boldsymbol{P}}_N), \quad \tilde{\boldsymbol{R}} = \text{block-diag}\,(\tilde{\boldsymbol{R}}_1, \ldots, \tilde{\boldsymbol{R}}_N), \quad \boldsymbol{U} = \begin{pmatrix} \boldsymbol{U}_1 & \ldots & \boldsymbol{U}_N \end{pmatrix}, \quad \boldsymbol{V} = \begin{pmatrix} \boldsymbol{V}_1 \\ \vdots \\ \boldsymbol{V}_N \end{pmatrix}$$

  - Reduction conditions on $\tilde{g}$ ensured by conditions on matrix data $(\tilde{\boldsymbol{P}}, \tilde{\boldsymbol{R}}, \boldsymbol{U}, \boldsymbol{V})$, these conditions are fulfilled because they hold for each block $(\tilde{\boldsymbol{P}}_i, \tilde{\boldsymbol{R}}_i, \boldsymbol{U}_i, \boldsymbol{V}_i)$
- $m = 3$: Hyperextreme multi-Demiański-Newman solution (charged generalization of multi-Kerr-NUT) obtained analogously

Conclusions

- Generalization of solution generating method to non-constant $\boldsymbol{P}, \boldsymbol{R}$
  - Necessary for application to non-autonomous chiral model
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  - Reductions to well known multi-Kerr-NUT and hyperextreme multi-Demiański-Newman solutions
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- Method produces much larger class of solutions
  - Solutions with non-diagonal $\tilde{\boldsymbol{P}}, \tilde{\boldsymbol{R}}$ known (some are limits of solutions with diagonal $\tilde{\boldsymbol{P}}, \tilde{\boldsymbol{R}}$, no systematic exploration yet)
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- Non-autonomous chiral model and Ernst equation addressed in new way using bidifferential calculus
  - Established solid foundations, but still much to explore!