NBI LECTURES ON SCATTERING AMPLITUDES IN $N=0$ & $N=4$ GAUGE THEORY

PLAN:

I. TREE LEVEL GLUON TREES: COLOR DECOMPOSITION, SPINOR HELICITY FORMALISM

BCFW RECURSION

II. SYMMETRIES: CONFORMAL SYMMETRY

$N=4$ SUPER YANG-MILLS: SUPERAMPLITUDES

DUAL SUPERCONFORMAL SYMMETRY $\to$ YANGIAN SYMMETRY

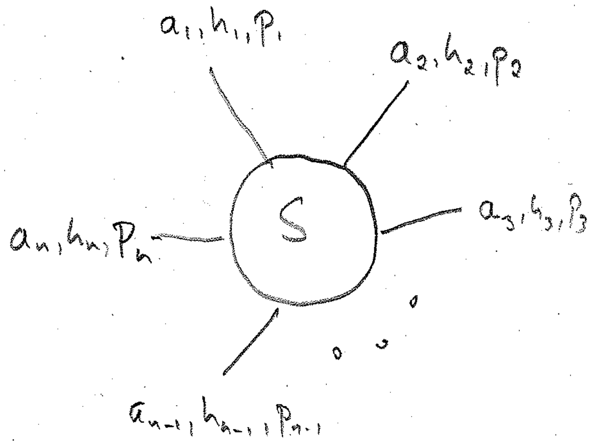
III. 1-LOOP: GENERALIZED UNITARITY RELATIONS

MHV-AMPLITUDES & RELATION TO LIGHT-LIKE

WILSON LOOPS

DUAL CONFORMAL ANOMALY, REMAINDER FUNCTION.

I. TREE LEVEL AMPLITUDES



Interested in gluon scattering amplitudes:

Quantum numbers:

- Colour: $a_i = 1, 2, \ldots, N^2-1$

  (SU(N) gauge theory)

- Helicity: $h_i = \pm 1$

- Momentum: p_i with $p_i^2 = 0$ (massless)

Textbook approach: Feynman diagrams, but increasingly complex for large n or large # of loops.

E.g. $n=6$ tree-level: graphs.

I.1 COLOR DECOMPOSITION

Can decouple color structure from kinematics: Partial or color ordered amps

$$\mathcal{A}_n^{\text{tree}}(\{a_i,h_i,p_i\}) = g^{n-2} \sum_{\sigma\in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma_1}}\cdots T^{a_{\sigma_n}}) \times A_n^{\text{tree}}(\sigma_1^{h_{\sigma_1}}\,\sigma_2^{h_{\sigma_2}}\ldots\sigma_n^{h_{\sigma_n}})$$

g: gauge coupling, $\alpha_s = \frac{g^2}{4\pi}$

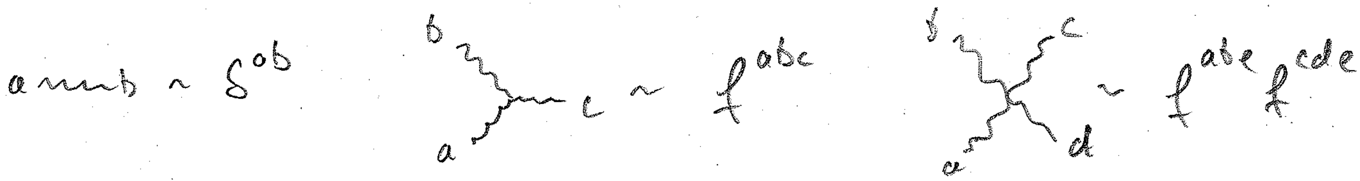
How?

$(T^a)_i{}^j$: traceless $N\times N$ hermitian matrices.

$$\mathrm{tr}(T^aT^b) = \delta^{ab} \qquad [T^a,T^b] = i\sqrt{2}\, f^{abc}\, T^c$$

↖ structure consts of SU(N).

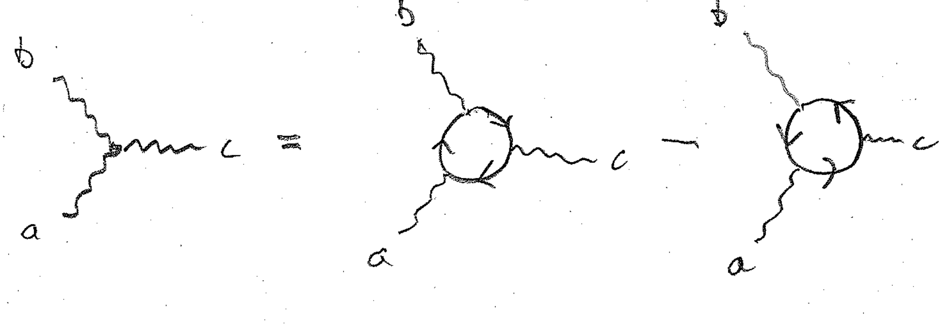
□ Color structure of Feynman rules:



$a \sim b \sim \delta^{ab}$ $\quad\sim f^{abc}\quad\sim f^{abe}f^{cde}$

$$f^{abc} = -\frac{i}{\sqrt{2}}\left[\mathrm{Tr}(T^aT^bT^c) - \mathrm{Tr}(T^aT^cT^b)\right]$$

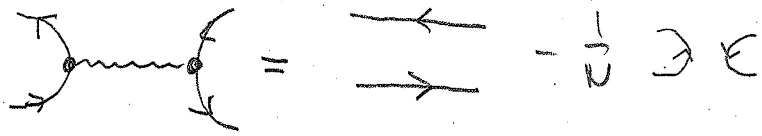
Graphical repr:



SU(N) identity:

$$(T^a)_{i_1}{}^{j_1}(T^a)_{i_2}{}^{j_2} = \delta_{i_1}{}^{j_2}\,\delta^{j_1}{}_{i_2} - \frac{1}{N}\,\delta_{i_1}^{j_1}\,\delta_{i_2}^{j_2}$$

Graphical repr:



$\Rightarrow$ Any tree-diagram reduces to a single trace structure

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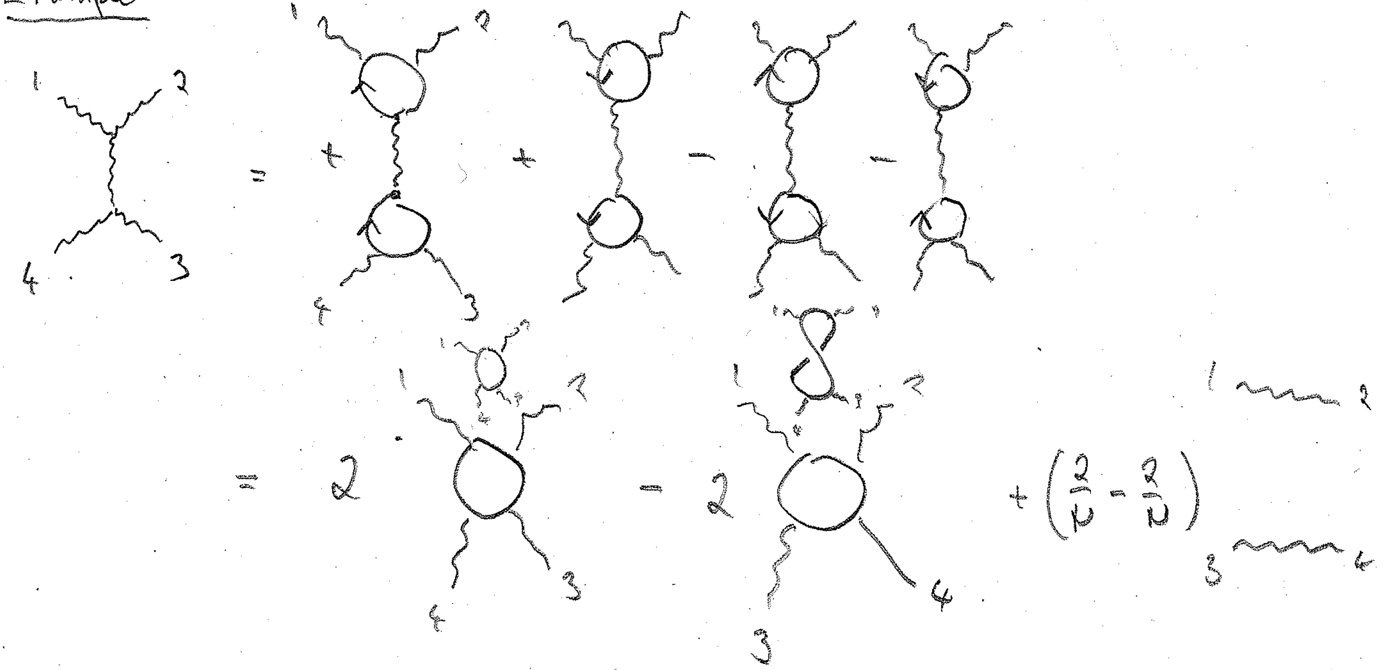
Example:



$$= \begin{pmatrix} \mathrm{Tr}(1234) \\ + \mathrm{Tr}(1432) \end{pmatrix} A(1,2,3,4) - \begin{pmatrix} \mathrm{Tr}(1243) \\ + \mathrm{Tr}(1342) \end{pmatrix} A(1243)$$

+ t & u channel.

UPSHOT:

Compute all colour ordered or partial amplitudes A_n^{tree}.

- Are diagrams with particular cyclic ordering of gluons
- Poles can only occur in cyclically adjacent momenta.

I.2 GENERAL PROPERTIES OF COLOR ORDERED AMPLITUDES:

- CYCLICITY: $A(1,2,\ldots n) = A(2,3,\ldots n,1)$

- REFLECTION: $A(1,2,\ldots,n) = (-)^n A(n,\ldots,1)$

  (from a/sym of color ordered Feynman rules)

- PHOTON DECOUPLING:

  Colour decomposition also valid for $U(N)$ gauge group.

  Add $(T^0)_i{}^j = \frac{1}{\sqrt{N}} \delta_j^i$ to $(T^a)^i{}_j$ $a = 1,\ldots,N^2-1$

  But photon doesnt couple to gluons: $f^{0ab} = 0$,

  $\Rightarrow A_n^{tree}(\{a_i, h_i, p_i\}, 0, h_0, p_0) = 0$

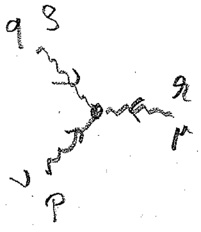
  Insert identity matrix into colour decomposition formula: All terms with same colour structure sum upto zero

  $\Rightarrow A_n^{tree}(1,2,3\ldots,n,0) + A_n^{tree}(1,0,2,3,\ldots,n) + A_n^{tree}(1,2,0,3,\ldots,n)$

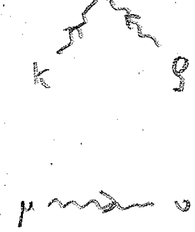
  $+ A_n^{tree}(1,2,\ldots,0,n) = 0.$

- PARITY: Helicity flip of all legs leaves amp. invariant.

COLOR ORDERED FEYNMANRULES



$$= \frac{i}{\sqrt{2}} \left[\eta_{\nu\rho} (p-q)_\mu + \eta_{\rho\mu} (q-r)_\nu + \eta_{\mu\nu} (r-p)_\rho \right]$$



$$= \frac{i}{2} \left[2\, \eta_{\mu\rho} \eta_{\nu\kappa} - \eta_{\mu\nu} \eta_{\rho\kappa} - \eta_{\mu\kappa} \eta_{\nu\rho} \right]$$

$$= -\frac{i}{p^2} \eta_{\mu\nu}$$

I.3 SPINOR HELICITY FORMALISM

- Write 4-momentum p^μ as bispinor $(\alpha = 1,2;\ \dot\alpha = 1,2)$

$$p^\mu \to p^{\alpha\dot\alpha} = \sigma_\mu^{\alpha\dot\alpha}\, p^\mu \qquad \sigma_\mu = (\mathbb{1}, \vec\sigma)$$

Then

$$p_i^{\alpha\dot\alpha}\, p_j^{\beta\dot\beta}\, \varepsilon_{\alpha\beta}\, \varepsilon_{\dot\alpha\dot\beta} = p_i^\mu\, p_j^\nu \underbrace{(\sigma_\mu)_{\alpha\dot\alpha}\, (\sigma_\nu)^{\alpha\dot\alpha}}_{2\eta_{\mu\nu}} = 2\, p_i \cdot p_j$$

- Mass-shell condition:

$$p_i^2 = 0 \iff \det(p_i^{\alpha\dot\alpha}) = 0 \iff \boxed{p_i^{\alpha\dot\alpha} = \lambda_i^\alpha\, \tilde\lambda_i^{\dot\alpha}}$$

- Helicity spinors λ_i^α and $\tilde\lambda_i^{\dot\alpha}$, with $(\lambda_i^\alpha)^* = \pm\, \tilde\lambda_i^{\dot\alpha}$

where sign is determined by sign of energy component p^0.

Note:

λ_i & $\tilde\lambda_i$ not unambiguously fixed by $p_i^{\alpha\dot\alpha} = \lambda_i^\alpha\, \tilde\lambda_i^{\dot\alpha}$ as rescalings

$$\lambda \to e^{i\phi}\lambda \quad ; \quad \tilde\lambda \to e^{-i\phi}\tilde\lambda$$

leave p_i invariant.

These rescalings are generated by the helicity operator:

$$h = \frac{1}{2}\left[-\lambda^\alpha \frac{\partial}{\partial\lambda^\alpha} + \tilde\lambda^{\dot\alpha} \frac{\partial}{\partial\tilde\lambda^{\dot\alpha}} \right]$$

Convention: λ: helicity $-1/2$, $\tilde\lambda$: helicity $+1/2$.

- From $\{\lambda_i, \tilde\lambda_i\}$ can build the SO(2) invariants

$$\langle \lambda_i \lambda_j \rangle := \lambda_i^{\alpha} \lambda_{j\alpha} = - \langle \lambda_j \lambda_i \rangle = \langle i j \rangle \quad \text{for short}$$

$$[\lambda_i \lambda_j] := \tilde\lambda_{i\dot\alpha} \tilde\lambda_j^{\dot\alpha} = - [\lambda_j \lambda_i] = [i j]$$

- Then $2\, p_i \cdot p_j = p_i^{\alpha\dot\alpha} p_{j\alpha\dot\alpha} = \langle ij \rangle [ji] = (p_i + p_j)^2 = s_{ij}$

(Mandelstam invariants).

- For positive energy states $p_i^0 > 0$, $p_j^0 > 0$ one shows

$$\langle ij \rangle = \sqrt{|s_{ij}|}\, e^{i\phi_{ij}} \qquad [ij] = - e^{-i\phi_{ij}} \sqrt{|s_{ij}|}$$

$$\cos\phi_{ij} = \frac{p_i^1 p_j^+ - p_j^1 p_i^+}{\sqrt{|s_{ij}|\, p_i^+ p_j^+}} \qquad \boxed{p^{\pm} = p^0 \pm p^3}$$

$$\sin\phi_{ij} = p_i^2 p_j^+ - p_j^2 p_i^+ \Big/ \sqrt{|s_{ij}|\, p_i^+ p_j^+} \qquad \boxed{\tan\phi_{ij} = \frac{p_i^2 p_j^+ - p_j^2 p_i^+}{p_i^1 p_j^+ - p_j^1 p_i^+}}$$

- <u>Identities</u>:

i) Schouten: $\langle \lambda_1 \lambda_2 \rangle \lambda_3^{\alpha} + \langle \lambda_2 \lambda_3 \rangle \lambda_1^{\alpha} + \langle \lambda_3 \lambda_1 \rangle \lambda_2^{\alpha} = 0$

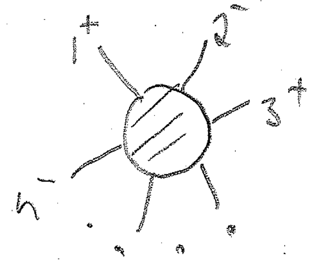
or $\langle 12 \rangle \langle 34 \rangle + \langle 23 \rangle \langle 14 \rangle + \langle 31 \rangle \langle 24 \rangle = 0.$

ii) Total momentum conservation in scattering amplitudes:

$$\sum_{i=1}^{n} p_i^{\mu} = 0 \iff \sum_{i=1}^{n} \lambda_i^{\alpha} \tilde\lambda_i^{\dot\alpha} = 0 \iff \sum_{i=1}^{n} \langle a | i \rangle [i | b] = 0$$

POLARIZATION

Partial amps depend on momenta and helicities ± 1 of external gluons:



$$A_n := \varepsilon_+^{\mu_1} \varepsilon_-^{\mu_2} \varepsilon_+^{\mu_3} \cdots \varepsilon_-^{\mu_n} \, A_{\mu_1 \ldots \mu_n}(p_1 \ldots p_n)$$

Polarization vector $\varepsilon_\pm^\mu$ in terms of helicity spinors:

$$\varepsilon_{+,i}^{\alpha\dot\alpha} = -\sqrt{2} \, \frac{\tilde\lambda_i^{\dot\alpha} \mu_i^\alpha}{\langle \lambda_i \mu_i \rangle}$$

$$\varepsilon_{-,i}^{\alpha\dot\alpha} = \sqrt{2} \, \frac{\lambda_i^\alpha \tilde\mu_i^{\dot\alpha}}{[\tilde\lambda_i \tilde\mu_i]}$$

with arbitrary reference momentum $\xi_i^{\alpha\dot\alpha} = \mu_i^\alpha \tilde\mu_i^{\dot\alpha}$.

☐ Amplitude does not depend on choice of ξ_i (gauge transformation):

Shift $\mu \to \mu + \delta\mu$

$$\delta\varepsilon_+^{\alpha\dot\alpha} = -\sqrt{2} \left(\frac{\tilde\lambda^{\dot\alpha} \delta\mu^\alpha}{\langle \lambda\mu \rangle} - \tilde\lambda^{\dot\alpha} \mu^\alpha \frac{\langle \lambda\, \delta\mu \rangle}{\langle \lambda\mu \rangle^2} \right)$$

$$= -\sqrt{2} \, \frac{1}{\langle \lambda\mu\rangle^2} \tilde\lambda^{\dot\alpha} \left(\delta\mu^\alpha \langle \lambda\mu \rangle - \mu^\alpha \langle \lambda\, \delta\mu \rangle \right)$$

$$= -\sqrt{2} \, \lambda^\alpha \tilde\lambda^{\dot\alpha} \frac{\langle \mu\, \delta\mu \rangle}{\langle \lambda\mu \rangle^2} \sim p^{\alpha\dot\alpha}$$

Hence

$$\delta\varepsilon_+^\mu(p) \, A_{\mu\nu_1 \ldots \nu_n}(p, q_1 \ldots q_{n-1}) = \delta\varepsilon_+^\mu(p) \, \langle A_\mu(p) \, A_{\nu_1}(q_1) \ldots A_{\nu_n}(q_n) \rangle$$

$$\sim p^\mu \langle A_\mu(p) \ldots \rangle = 0 .$$

→ Amplitudes depend on $\{\lambda_i, \tilde\lambda_i, \pm_i\}$ only.

I.4 Helicity classification of gluon amps

Classify gluon amps according to # of negative helicity gluons:

$$A_n^{tree}(1^+, 2^+, \dots, n^+) = 0$$

$$A_n^{tree}(1^-, 2^+, \dots, n^+) = 0$$

will show later, can be seen by suitable choice of ξ_i vectors.

First non-trivial case: 2 neg. helicity states at positions i & j:

MHV-amplitude ("maximally helicity violating")

$$\boxed{A_n^{MHV}(i^-, j^-) = i\, \delta^{(4)}\Big(\sum_{i=1}^n p_i\Big) \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}}$$

Parke-Taylor, '86

Will prove soon. Note "holomorphic" structure.

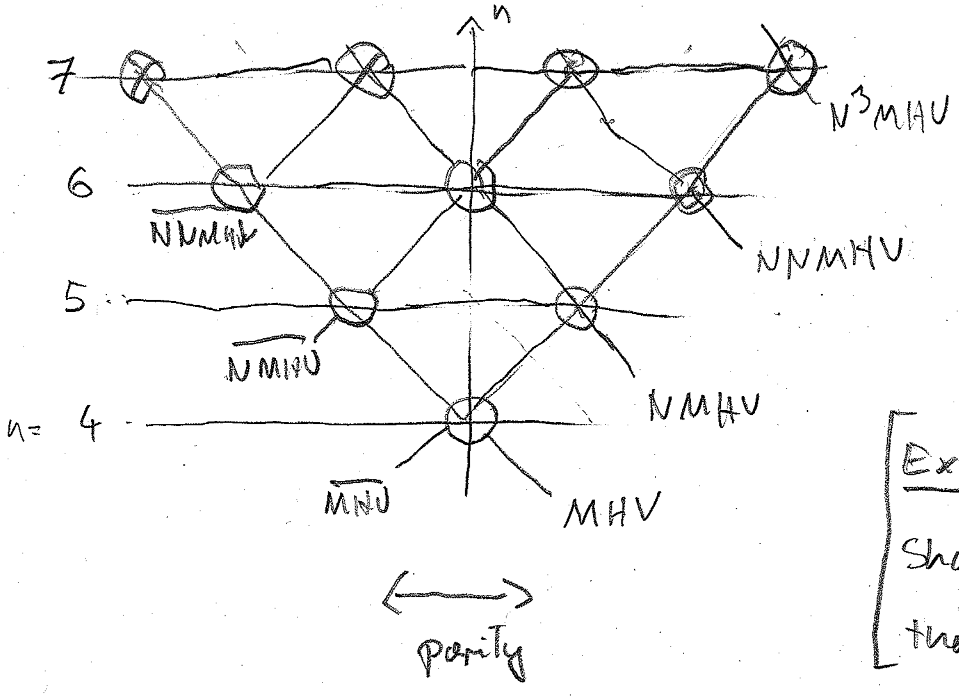
N.B. parity implies $A_n^{tree}(1^-, \dots, n^-) = A_n^{tree}(1^+, 2^-, \dots, n^-) = 0$

$$A_n^{\overline{MHV}}(1^- \dots i^+ \dots j^+ \dots n^-) = i\, \delta^{(4)}(p) \frac{[ij]^4}{[12][23]\dots[n1]}$$

Classification:

Amp with $(p+2)$ negative helicity gluons: $(\text{Next-to})^p$ MHV amps.

N^pMHV.



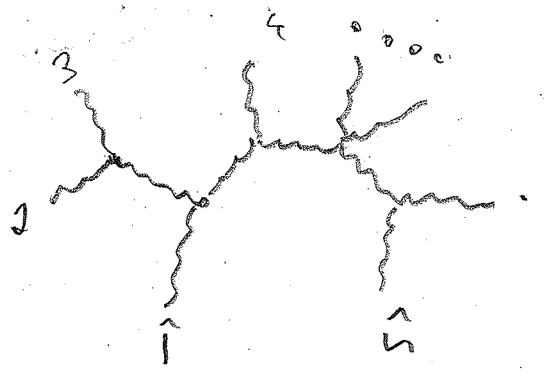
[Ex:

Show for $n=4$

that $A_4^{MHV} = A_4^{\overline{MHV}}$.]

I.5 BCFW ONSHELL RECURSION

Consider n-gluon tree amplitude



Perform the complex shift: $z \in \mathbb{C}$

$$\lambda_1 \to \hat{\lambda}_1(z) = \lambda_1 - z\lambda_n$$

$$\tilde{\lambda}_n \to \hat{\tilde{\lambda}}_n(z) = \tilde{\lambda}_n + z\tilde{\lambda}_1$$

this leads to

$$p_1^{\alpha\dot{\alpha}} \to \hat{p}_1^{\alpha\dot{\alpha}}(z) = (\lambda_1 - z\lambda_n)^\alpha \tilde{\lambda}_1^{\dot{\alpha}}$$

$$p_n^{\alpha\dot{\alpha}} \to \hat{p}_n^{\alpha\dot{\alpha}}(z) = \lambda_n^\alpha (\tilde{\lambda}_n + z\tilde{\lambda}_1)^{\dot{\alpha}}$$

□ Shift preserves on-shellness and total momentum conservation:

$$\hat{p}_1(z)^2=0 \qquad \hat{p}_n(z)^2=0 \qquad \hat{p}_1(z)+\hat{p}_n(z)=p_1+p_n$$

□ Q: What are the analytic properties of the deformed amplitude $A_n(z)$?

$$A_n(z)=\delta^{(4)}\Big(\sum_i p_i\Big)\,\mathcal{A}_n(z)$$

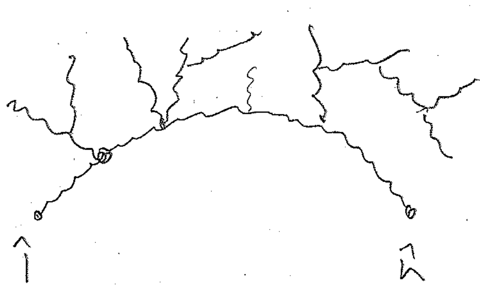
$\mathcal{A}_n(z)$ is rational function of $\{\lambda_i,\tilde{\lambda}_i\}$ and z. As $\mathcal{A}_n(z=0)$ only has local poles in region momenta

$$\frac{1}{(p_i+p_{i+1}+\dots+p_j)^2}=:\frac{1}{P_i}$$

the deformed amplitude $\mathcal{A}_n(z)$ will have only simple poles in z of the form:

$$\frac{1}{\hat{P}_i(z)^2}=\frac{1}{(\hat{p}_1(z)+p_2+\dots+p_{i-1})^2}=\frac{1}{P_i^2-z\langle n|P_i|1]}$$

with $P_i:=p_1+\dots+p_{i-1}$ and $\langle n|P|1]:=\lambda_n^{\alpha}P_{\alpha\dot{\alpha}}\tilde{\lambda}_1^{\dot{\alpha}}$.



$\Rightarrow$ $\mathcal{A}_n(z)$ has a simple pole at $z_{P_i}=\dfrac{P_i^2}{\langle n|P_i|1]}$

$\forall\, i\in[2,n-1]$.

Near the pole z_p the amplitude $\mathcal{A}_n(z)$ behaves as

$$\lim_{z \to z_p} \mathcal{A}_n(z) \sim \frac{1}{z - z_p} \frac{-1}{\langle n | P | 1]} \sum_{S} \mathcal{A}_L^S(\hat{1}(z_p), 2, \ldots, i-1, -\hat{P}(z_p)) \, \mathcal{A}_R^{\bar{S}}(\hat{P}(z_p), i, \ldots, \hat{n}(z_p))$$

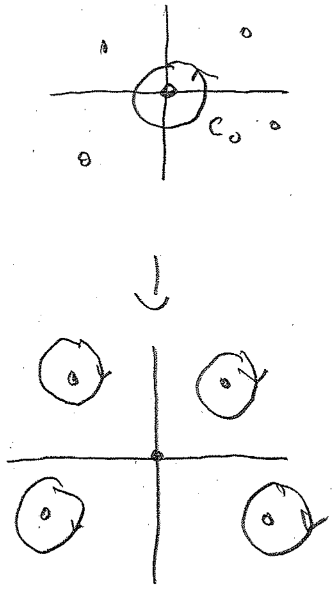
- Internal propagator goes on-shell : Factorization of $\mathcal{A}_n$ into L and R on-shell subamplitudes
- Sum over S : Sum over all d.o.f. For gluons $S = \{+, -\}$.

Now consider fct. $\frac{\mathcal{A}(z)}{z}$. This behaves as

$$\lim_{z \to z_p} \frac{\mathcal{A}(z)}{z} \sim -\frac{1}{z - z_p} \sum_{S} \mathcal{A}_L^S(z_p) \frac{1}{P^2} \mathcal{A}_R^{\bar{S}}(z_p)$$

The initial amplitude we are interested in is

$$\mathcal{A}_n = \mathcal{A}_n(0) = \oint_{C_0} \frac{dz}{2\pi i} \frac{\mathcal{A}(z)}{z}$$



$$= \sum_{i} \sum_{S} \mathcal{A}_L^S(z_p) \frac{1}{P_i^2} \mathcal{A}_R^{\bar{S}}(z_p)$$

$$+ \operatorname{Res}(z = \infty)$$

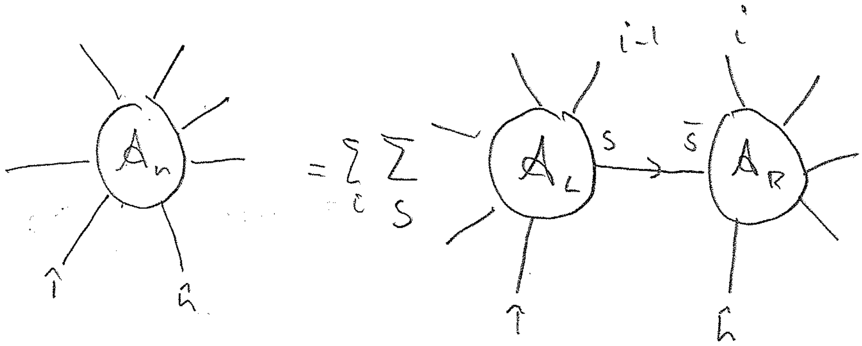
Will see that $\operatorname{Res}(z=\infty)$ vanishes for gauge theories.

Assuming this we have

BCFW, '05.

$$\Rightarrow \boxed{A_n(0) = \sum_{i=2}^{n-1} \sum_{s \in \{\pm\}} A_L^s\left(\hat{1}(z_{P_i}), 2, \ldots, i-1, -\hat{P}_i(z_{P_i})\right) \frac{1}{P_i^2} A_R^{\bar{s}}\left(\hat{P}(z_{P_i}), i, \ldots, \hat{n}(z_{P_i})\right)}$$

On-shell recursion relation.



A_L: i legs

A_R: n-i+1 legs

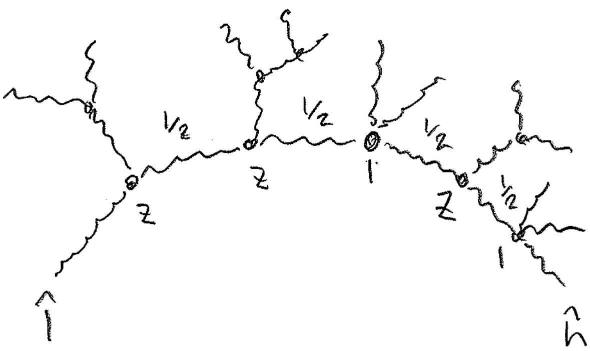
Constructive: Can build higher point amps from lower ones.

Comments:
- We chose neighboring legs $\hat{1}$ & $\hat{n}$ for shifts. Not necessary.
- Also multiple shifts of more than 2 legs have been considered.

Large z behaviour of $A(z)$

Recall: $\oint dz \frac{A(z)}{z} \Rightarrow$ need falloff $\lim_{z\to\infty} A(z) \sim \frac{1}{z}$

Generic graph ($\hat{1}$ & $\hat{n}$ are neighbours): z-estimates:



$\sim z$ (at most).

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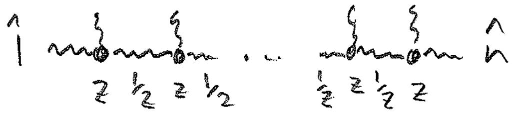
$\sim \frac{1}{z}$ for line from $\hat{1}$ to $\hat{n}$.

- For line $\hat{1} \to \hat{n}$ worst case scenario is from only 3-valent vertices



$$\sim z$$

- Scaling also depends on helicity of shifted legs:

Scalings of polarization vectors:

$$(\varepsilon_1^+)^{\alpha\dot\alpha} = -\sqrt{2}\,\frac{\tilde\lambda_1^{\dot\alpha}\,\mu^\alpha}{\langle\hat\lambda_1(z)\,\mu\rangle} \sim \frac{1}{z} \qquad (\varepsilon_n^+)^{\alpha\dot\alpha} = -\sqrt{2}\,\frac{\hat{\tilde\lambda}_n(z)^{\dot\alpha}\,\mu^\alpha}{\langle\lambda_n,\mu\rangle} \sim z$$

$$(\varepsilon_1^-)^{\alpha\dot\alpha} = \sqrt{2}\,\frac{\hat\lambda_1(z)^{\alpha}\,\tilde\mu^{\dot\alpha}}{[\tilde\lambda_1\,\tilde\mu]} \sim z \qquad (\varepsilon_n^-)^{\alpha\dot\alpha} = \sqrt{2}\,\frac{\lambda_n(z)^{\alpha}\,\tilde\mu^{\dot\alpha}}{[\hat{\tilde\lambda}_n(z)\,\tilde\mu]} \sim \frac{1}{z}$$

Summary: At worst

$$\boxed{A(\hat{1}^+\,\hat{n}^-) \sim \frac{1}{z} \quad \checkmark} \qquad A(\hat{1}^+\,\hat{n}^+) \sim z$$

$$A(\hat{1}^-\,\hat{n}^-) \sim z \qquad A(\hat{1}^-\,\hat{n}^+) \sim z^3$$

Always possible to use $\underline{\{\hat{1}^+, \hat{n}^-\}}$ shift and a valid on-shell recursion.

Can show (closer look): Also $(+,+)$ and $(-,-)$ shifts lead to $1/z$ scalings.

$\Rightarrow$ $\boxed{\text{Shifted legs may be } (1,n) = \{(+,-), (+,+), (-,-)\} \text{ but not } (-,+)}$

Exercise 1: $\quad MHV_4 = \overline{MHV}_4$

Tip: Use momentum conservation to prove rels like

$$\frac{\langle 12\rangle}{\langle 41\rangle} = \frac{[43]}{[23]} \qquad \frac{\langle 12\rangle}{\langle 34\rangle} = \frac{[12]}{[34]}$$

$$\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \overset{!}{=} \frac{[34]^4}{[12][23][34][41]}$$

Momentum conservation $\quad \langle 12\rangle[23] + \langle 14\rangle[43] = 0 \Rightarrow \dfrac{\langle 12\rangle}{\langle 41\rangle} = \dfrac{[43]}{[23]}$

$1 \leftrightarrow 2$, $4 \leftrightarrow 3$: $\quad \dfrac{\langle 21\rangle}{\langle 32\rangle} = \dfrac{[34]}{[14]}$

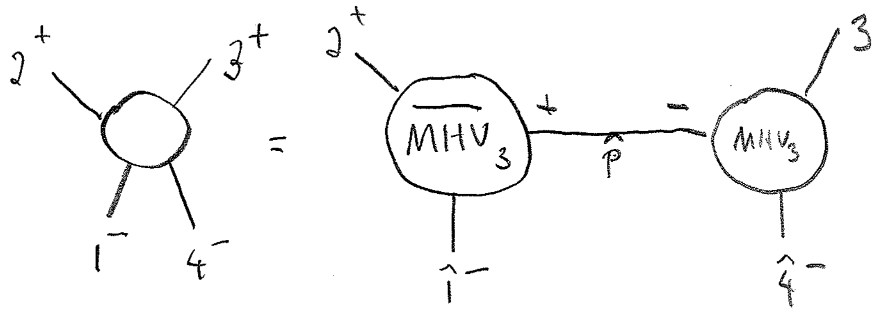
$$\Rightarrow MHV_4 = \frac{\langle 12\rangle}{\langle 23\rangle}\frac{\langle 12\rangle}{\langle 41\rangle}\frac{\langle 12\rangle}{\langle 34\rangle} = \frac{[34]^2}{[41][23]}\frac{\langle 12\rangle}{\langle 34\rangle} = \frac{[34]^3}{[12][23][41]}$$

$p_1 \cdot p_2 = p_3 \cdot p_4 \Rightarrow \langle 12\rangle[21] = \langle 34\rangle[43]$

$$\Rightarrow \frac{\langle 12\rangle}{\langle 34\rangle} = \frac{[43]}{[21]} = \frac{[34]}{[12]}$$

Exercise 2:

Proof of 4pt MHV amplitude from BCFW.



$$= \frac{[2\hat{P}]^3}{[\hat{1}2][\hat{P}\hat{1}]}\,\frac{1}{\langle 12\rangle[21]}\,\frac{\langle \hat{4}\hat{P}\rangle^3}{\langle \hat{P}3\rangle\langle 3\hat{4}\rangle} =$$

$$\lambda_1 \to \lambda_1 - z\lambda_4$$
$$\tilde{\lambda}_4 \to \tilde{\lambda}_4 + z\tilde{\lambda}_1$$
$$\Rightarrow [\hat{1}\,*] = [1\,*]$$
$$\Rightarrow \langle \hat{4}\,*\rangle = \langle 4\,*\rangle$$

$$= \frac{[2\hat{P}]^3}{[12][\hat{P}1]}\,\frac{1}{\langle 12\rangle[21]}\,\frac{\langle 4\hat{P}\rangle^3}{\langle \hat{P}3\rangle\langle 34\rangle}$$

I.6 THE 3-POINT AMPLITUDE

For real momenta there is no 3-pt amplitude for massless particles:

$$p_1^\mu + p_2^\mu + p_3^\mu = 0 \qquad \Rightarrow \qquad p_1 \cdot p_2 = p_2 \cdot p_3 = p_3 \cdot p_1 = 0$$

Different situation if we allow $p_i^\mu \in \mathbb{C}$:

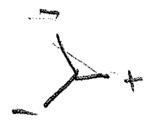
$$p_i \cdot p_j = 0 \quad \Rightarrow \quad \langle i j \rangle [i j] = 0 \quad \Rightarrow \quad \text{either } \langle i j \rangle = 0 \quad \text{or } [i j] = 0$$

$$\forall\, i,j \in \{1,2,3\}.$$

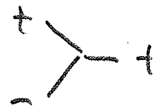
Hence either $\lambda_1^\alpha \propto \lambda_2^\alpha \propto \lambda_3^\alpha$ or $\tilde{\lambda}_1^{\dot\alpha} \propto \tilde{\lambda}_2^{\dot\alpha} \propto \tilde{\lambda}_3^{\dot\alpha}$.
For complex momenta no relation between λ_i and $\tilde{\lambda}_i$.

2 choices:

MHV_3 : $[12] = [23] = [31] = 0$



$\overline{\text{MHV}}_3$: $\langle 12 \rangle = \langle 23 \rangle = \langle 31 \rangle = 0$



Explicitly:

$$\boxed{\begin{aligned} A_3^{\text{MHV}}(i^-, j^-) &= i\, \frac{\langle i j \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \\ A_3^{\overline{\text{MHV}}}(i^+, j^+) &= i\, \frac{[i j]^4}{[12][23][31]} \end{aligned}}$$

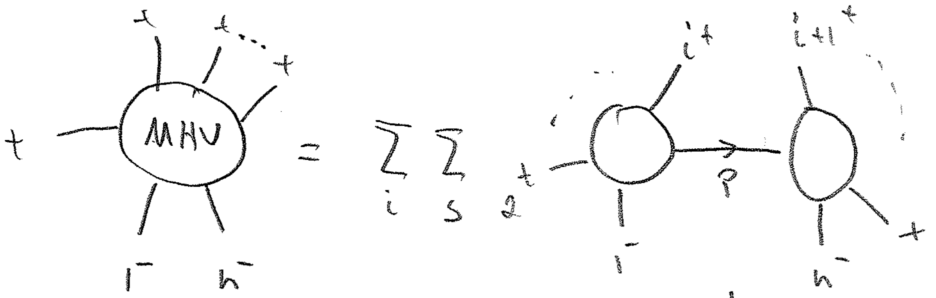
$$h_i = \tfrac{1}{2}\left[-\lambda_i \frac{\partial}{\partial \lambda_i} + \tilde{\lambda}_i \frac{\partial}{\partial \tilde{\lambda}_i}\right]$$

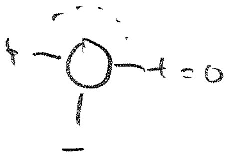
Form consistent with helicities:

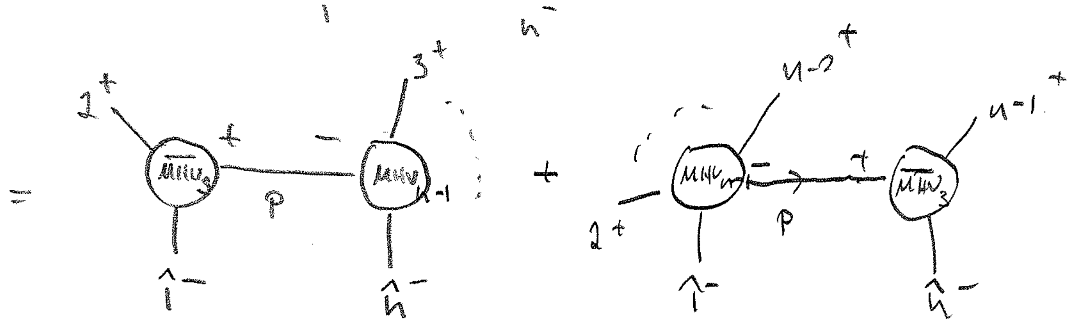
$$h_i A_3^{\text{MHV}} = (4-2) \cdot \left(-\tfrac{1}{2}\right) A_3^{\text{MHV}} = (-1) A_3$$

$$h_{\text{not}(i,j)} A_3^{\text{MHV}} = (-2) \cdot \left(-\tfrac{1}{2}\right) A_3 = (1)\, A_3^{\text{MHV}} \qquad h_j A_3^{\text{MHV}} = (-1)\, A_3^{\text{MHV}}$$

Apply BCFW recursion for MHU-amplitudes



as  $=0$



Now recall shifts:
$$\lambda_1 \to \lambda_1 - z\lambda_n$$
$$\tilde\lambda_n \to \tilde\lambda_n + z\tilde\lambda_1$$

Subtle point:

$\overline{MHV}_3$ vertex at A_R not allowed:

$\overline{MHV}_3$ prescription implies $0 \overset{!}{=} \langle \hat n, n-1\rangle = \langle n, n-1\rangle$

$\Rightarrow\ p_n \cdot p_{n-1} = 0$ not generally true

Not a problem for $\overline{MHV}_3$ at A_L as $\langle \hat 1 2\rangle \neq \langle 12\rangle$.

Exercise 2:

Prove the Park-Taylor formula using BCFW relation

II. SYMMETRIES OF TREE-LEVEL AMPLITUDES

II.1 CONFORMAL SYMMETRY

Scattering amplitudes are invariant under Poincaré-transformations.

Obvious symmetries:

$$\boxed{p^{\alpha\dot\alpha} = \sum_{i=1}^{n} \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha}}\,, \qquad \boxed{m_{\alpha\beta} = \sum_{i=1}^{n} \lambda_{i(\alpha}\, \partial_{i\beta)}}\,, \qquad \boxed{\bar m_{\dot\alpha\dot\beta} = \sum_{i=1}^{n} \tilde\lambda_{i(\dot\alpha}\, \partial_{i\dot\beta)}}$$

momentum — Lorentz

with $\partial_{\dot\alpha} = \frac{\partial}{\partial\tilde\lambda^{\dot\alpha}}$, $\partial_\alpha = \frac{\partial}{\partial\lambda^\alpha}$.

Obey: $$\left\{ \begin{aligned} &p^{\alpha\dot\alpha} A_n(\{\lambda_i,\tilde\lambda_i\}) = 0 \\ &\bar m_{\dot\alpha\dot\beta} A_n = m_{\alpha\beta} A_n = 0 \end{aligned} \right\}$$ in distributional sense ($p\,\delta(p) = 0$)

Yang-Mills Theory is classically conformal invariant ($\mathcal{N}=4$ SYM also at the quantum level):

$\Rightarrow$ Tree-level amplitudes should be conformal invariant:

$$k_{\alpha\dot\alpha} A_n(\{\lambda_i,\tilde\lambda_i\}) = 0$$

$$\wedge \quad d\, A_n(\{\lambda_i\}) = 0$$

(less obvious symmetries)

Relevant representation: (Witten, 03):

$$\boxed{\begin{aligned} k_{\alpha\dot\alpha} &= \sum_{i=1}^{n} \partial_{i\alpha}\,\partial_{i\dot\alpha} \\ d &= \sum_{i=1}^{n} \left[\tfrac{1}{2}\lambda_i^\alpha \partial_{i\alpha} + \tfrac{1}{2}\tilde\lambda_i^{\dot\alpha} \partial_{i\dot\alpha} + 1\right] \end{aligned}}$$

Special conformal gen.

dilatation gen.

Ex: Show that $\left[k_{\alpha\dot\alpha}, p^{\beta\dot\beta}\right] = \delta_\alpha^\beta \delta_{\dot\alpha}^{\dot\beta}\, d + m_\alpha{}^\beta \delta_{\dot\alpha}^{\dot\beta} + \bar m_{\dot\alpha}{}^{\dot\beta} \delta_\alpha^\beta$

Invariance of MHV-amplitudes

$$A_n^{MHV} = g^{n-2}\, \delta^{(4)}\Big(\sum_i \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha}\Big) \frac{\langle \lambda_s, \lambda_t\rangle^4}{\langle 12\rangle \dots \langle n1\rangle}$$

① $d \circ A_n^{MHV}$: $\quad [\delta^{(4)}] = -4 \quad , \quad [\langle\lambda_s, \lambda_t\rangle] = 4$

$$\Rightarrow \quad [d, \delta^{(4)}(\textstyle\sum_i p_i)\, \langle s,t\rangle^4] = 0$$

$\dfrac{1}{\langle 12\rangle \dots \langle n1\rangle}$ is homogenous of degree -2 in all λ_i

$$\Rightarrow \quad d\, \frac{1}{\langle 12\rangle \dots \langle n1\rangle} = \sum_i \Big(\tfrac{1}{2}(-2) + 1\Big) \frac{1}{\langle 12\rangle \dots \langle n1\rangle} = 0$$

$$\Rightarrow \quad \boxed{d\, A_n^{MHV} = 0}$$

② $k_{\alpha\dot\alpha} A_n^{MHV} = \displaystyle\sum_i \frac{\partial^2}{\partial\lambda_{\alpha i}\, \partial\tilde\lambda_{i\dot\alpha}}\, \delta^{(4)}(P)\, A_n(\{\lambda_i\})$

$$= \sum_i \frac{\partial}{\partial\lambda_{\alpha_i}} \underbrace{\frac{\partial P^{\beta\dot\beta}}{\partial\tilde\lambda_{i\dot\alpha}}}_{\lambda_i^\beta \delta_{\dot\alpha}^{\dot\beta}} \left(\frac{\partial}{\partial p^{\beta\dot\beta}}\, \delta^{(4)}(P)\right) A_n(\{\lambda_i\})$$

$$= \left[\left(n \frac{\partial}{\partial p^{\alpha\dot\alpha}} + P^{\beta\dot\beta} \frac{\partial}{\partial p^{\beta\dot\alpha}} \frac{\partial}{\partial p^{\alpha\dot\beta}}\right) \delta^{(4)}(P)\right] A_n(\lambda_i)$$

$$+ \left(\frac{\partial}{\partial p^{\beta\dot\alpha}}\delta^{4}(P)\right) \sum_{i=1}^{n} \lambda_i^\beta \frac{\partial}{\partial\lambda_{\alpha i}}\, A_n(\{\lambda_i\})$$

<u>Use:</u> $$\sum_i \lambda_{i\alpha}\partial_{i\beta} = \sum_i \lambda_{i(\alpha}\partial_{i\beta)} + \frac{1}{2}\varepsilon_{\alpha\beta}\sum_i \lambda_i^\gamma \partial_{i\gamma}$$

$$\left[\begin{array}{ll} \lambda_1\partial_2 = \frac{1}{2}(\lambda_1\partial_2+\lambda_2\partial_1) - (\lambda^1\partial_1+\lambda^2\partial_2)\frac{1}{2} & \lambda^a = \varepsilon^{ab}\lambda_b \\ \lambda_2\partial_1 = \frac{1}{2}(\lambda_1\partial_2+\lambda_2\partial_1) + (\lambda^1\partial_1+\lambda^2\partial_2)\frac{1}{2} & \lambda^1 = \lambda_2\,,\ \lambda^2 = -\lambda_1 \end{array}\right.$$

$$\Rightarrow \sum_i \lambda_i^\gamma \partial_{i\beta} = \sum_i \varepsilon^{\gamma\alpha}\lambda_{i(\alpha}\partial_{i\beta)} - \frac{1}{2}\varepsilon^{\gamma\alpha}\varepsilon_{\alpha\beta}\sum_i \lambda_i^{\tilde\gamma}\partial_{i\tilde\gamma}$$

$$= \varepsilon^{\gamma\alpha}\underbrace{\sum_i \lambda_{i(\alpha}\partial_{i\beta)}}_{m_{\alpha\beta}} + \frac{1}{2}\delta^\gamma_\beta \sum_i \lambda_i^\delta \partial_{i\delta}$$

Thus:
$$\sum_{i=1}^n \lambda_i^\beta \frac{\partial}{\partial\lambda_{i\alpha}} A_n = \frac{1}{2}\delta^\beta_\alpha \sum_i \lambda_i^\delta \partial_{i\delta} A_n = -(n-4)\,\delta^\beta_\alpha A_n$$

$$\Rightarrow \quad k_{\alpha\dot\alpha} A_n^{MHV} = \left[4\frac{\partial}{\partial p^{\alpha\dot\alpha}}\delta^{(4)}(P) + P^{\beta\dot\beta}\frac{\partial}{\partial p^{\beta\dot\alpha}}\frac{\partial}{\partial p^{\alpha\dot\beta}}\delta^{(4)}(P)\right]$$

$$\times A_n^{MHV}(\{\lambda_i\})$$

Now in a distributional sense $P^{\beta\dot\beta}\frac{\partial}{\partial p^{\alpha\dot\beta}}\frac{\partial}{\partial p^{\dot\alpha\beta}}\delta^{(4)}(P) = -4\frac{\partial}{\partial p^{\alpha\dot\alpha}}\delta^{(4)}(P)$.

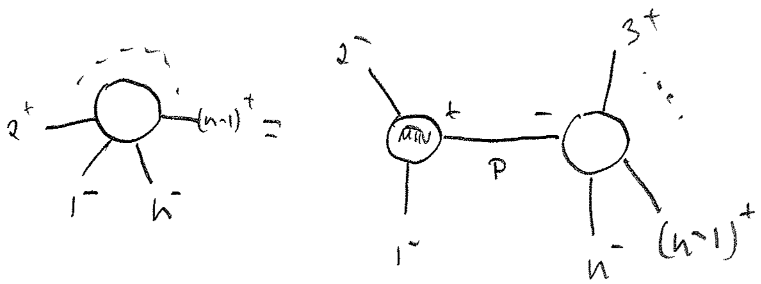
<u>Proof:</u>

$$\int d^4p\; F(p)\; P^{\beta\dot\beta}\frac{\partial}{\partial p^{\beta\dot\alpha}}\frac{\partial}{\partial p^{\alpha\dot\beta}}\delta^{(4)}(P)$$

$$= +\int d^4p\left[\frac{\partial}{\partial p^{\beta\dot\alpha}}F(p)\right]2\cdot\delta^\beta_\alpha + \int d^4p\left[\frac{\partial}{\partial p^{\alpha\dot\beta}}F(p)\right]2\,\delta^{\dot\beta}_{\dot\alpha}$$

$$= \int d^4p\; 4\left[\frac{\partial}{\partial p^{\alpha\dot\alpha}}F(p)\right]\delta^{(4)}(p) = \int d^4p\, F(p)\left(-4\frac{\partial}{\partial p^{\alpha\dot\alpha}}\delta^{(4)}(p)\right) \quad \square$$

Exercise 3: Proof of Parke-Taylor:



$[\hat{1} *] = [1 *]$
$\langle \hat{n} * \rangle = \langle n * \rangle$

$$= \frac{[2\hat{P}]^3}{[\hat{1}2][\hat{P}\hat{1}]} \; \frac{1}{\langle 12\rangle[21]} \; \frac{\langle \hat{n}\hat{P}\rangle^3}{\langle \hat{P}3\rangle\langle 34\rangle \dots \langle n-1, \hat{n}\rangle}$$

again
$$\langle n\hat{P}\rangle[\hat{P}2] = -\langle n\hat{1}\rangle[\hat{1}2] = -\langle n1\rangle[12]$$
$$\langle 3\hat{P}\rangle[\hat{P}1] = -\langle 32\rangle[21]$$

$$= \frac{\langle n1\rangle^3[12]^3}{[12][21]\;\langle 32\rangle[21]} \; \frac{1}{\langle 12\rangle\langle 34\rangle \dots \langle n-1, n\rangle} = -\frac{\langle n1\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 34\rangle \dots \langle n-1, n\rangle}$$

$$= -\frac{\langle 1, n\rangle^4}{\langle 12\rangle \dots \langle n1\rangle} \quad ✓.$$

II.2 N=4 SYM AMPLITUDES: SUPERAMPLITUDES

- N=4 SYM: 8+8 onshell states $A=1,2,3,4$ $SU(4)_R$-symmetry

bosons:	g_+	g_-	S_{AB}	fermions: Γ_A	$\bar{\Gamma}^A$
helicity	$+1$	-1	0	$1/2$	$-1/2$
# dof	1	1	6	4	4
	gluon		scalar	gluino	anti-gluino
$SU(4)_R$	singlet		a/sym (6)	fundamental (4)	anti-fundamental ($\bar{4}$)

- Assemble in single on-shell PCT-self conjugate supermultiplet:

Introduce: η^A $A=1,2,3,4$ (anti-commuting)

$$\Phi(\eta) = g_+ + \eta^A\Gamma_A + \frac{1}{2!}\eta^A\eta^B S_{AB} + \frac{1}{3!}\eta^A\eta^B\eta^C\varepsilon_{ABCD}\bar{\Gamma}^D + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\varepsilon_{ABCD}\, g_-$$

- Helicity: Assign helicity $\frac{1}{2}$ to η^A, then superfield Φ has helicity 1.

Helicity generator:
$$h := \frac{1}{2}\left[-\lambda^\alpha\partial_\alpha + \tilde{\lambda}^{\dot\alpha}\partial_{\dot\alpha} + \eta^A\frac{\partial}{\partial\eta^A}\right]$$
$\hookrightarrow \partial_A$.

$$h\,\Phi(\eta) = \Phi(\eta)$$

- Supersymmetry generators:

$$\boxed{p^{\alpha\dot\alpha} = \lambda^\alpha\tilde{\lambda}^{\dot\alpha} \qquad q^{\alpha A} = \lambda^\alpha\eta^A \qquad \bar{q}^{\dot\alpha}{}_A = \tilde{\lambda}^{\dot\alpha}\frac{\partial}{\partial\eta^A}}$$

□ Lorentz and R-symmetry generators:

$$\boxed{m_{\alpha\beta} = \lambda_{(\alpha}\partial_{\beta)} \qquad \bar{m}_{\dot\alpha\dot\beta} = \tilde\lambda_{(\dot\alpha}\partial_{\dot\beta)} \qquad r^A{}_B = \eta^A\partial_B - \tfrac{1}{4}\delta^A_B\,\eta^C\partial_C}$$

R-sym.

□ Superconformal generators

$$\boxed{k_{\alpha\dot\alpha} = \partial_\alpha\partial_{\dot\alpha} \qquad s_{\alpha A} = \partial_\alpha\partial_A \qquad \bar{s}_{\dot\alpha}{}^A = \eta^A\partial_{\dot\alpha}}$$

special conformal   super-conformal

$$\boxed{d = \tfrac{1}{2}\left[\lambda^\alpha\partial_\alpha + \tilde\lambda^{\dot\alpha}\partial_{\dot\alpha}\right]}$$

dilatation

They form the centrally extended $psu(2,2|4)$ symmetry algebra of $N=4$ SYM:

$$\{q^{\alpha A}, \bar{q}^{\dot\alpha}{}_B\} = \delta^A_B\, p^{\alpha\dot\alpha} \qquad \{s_{\alpha A}, \bar{s}_{\dot\alpha}{}^B\} = \delta^B_A\, k_{\alpha\dot\alpha}$$

$$\{q^{\alpha A}, s_{\beta B}\} = m^\alpha{}_\beta\,\delta^A_B + \delta^\alpha_\beta\, r^A{}_B + \tfrac{1}{2}\delta^\alpha_\beta\,\delta^A_B\,(d+c)$$

$$\{\bar{q}^{\dot\alpha}{}_A, \bar{s}_{\dot\beta}{}^B\} = \bar{m}^{\dot\alpha}{}_{\dot\beta}\,\delta^A_B - \delta^{\dot\alpha}_{\dot\beta}\, r^B{}_A + \tfrac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\,\delta^B_A\,(d-c)$$

$$[P,S] = \bar{Q} \qquad [K,Q] = \bar{S}$$

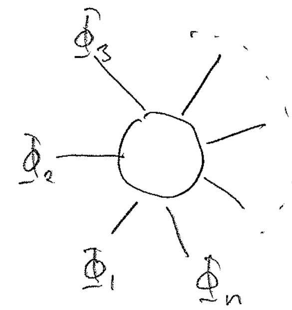
$$[K,P] = D + M + \bar{M}$$

□ Central charge:

$$c = 1 + \tfrac{1}{2}\left(\lambda^\alpha\partial_\alpha - \tilde\lambda^{\dot\alpha}\partial_{\dot\alpha} - \eta^A\partial_A\right) = 1 - h$$

Hence $c\,\Phi(\eta) = 0$.

□ We now consider <u>colour ordered superamplitudes</u>



$\mathbb{A}_n(\Phi_1, \Phi_2, \ldots, \Phi_n)$ which package all gluon-scalar-gluino amplitudes into one object.

The component amplitudes of interest emerge from η_i^A expansions.

$$\mathbb{A}_n(\Phi_1, \ldots, \Phi_n) = \mathbb{A}_n(\{\lambda_i^\alpha, \tilde{\lambda}_i^{\dot\alpha}, \eta_i^A\})$$

$$= (\eta_1)^4 (\eta_2)^4 A(-,-,+,\ldots,+) + (\eta_1)^4 (\eta_2)^3 (\eta_3) A(-,\bar\Gamma,\Gamma,+\ldots+) + \ldots$$

We have $\quad h_i \mathbb{A}(\Phi_1, \ldots, \Phi_n) = \mathbb{A}(\Phi_1, \ldots, \Phi_n)$ (local invariance!)

□ For superamplitudes $psu(2,2|4)$ symmetry generators are simply the sum of the single-particle representation from above:

$$P = \sum_{i=1}^n \lambda_i \tilde{\lambda}_i \quad , \quad q^{\alpha A} = \sum_{i=1}^n \lambda_i^\alpha \eta_i^A \quad , \quad \bar{q}^{\dot\alpha}{}_A = \sum_i \tilde{\lambda}_i^{\dot\alpha} \partial_{iA}$$

etc.

□ GENERAL FORM: (Super-poincaré invariance)

$$\boxed{\mathbb{A}_n(\Phi_1, \ldots, \Phi_n) = \frac{\delta^{(4)}(P)\, \delta^{(8)}(q)}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle} P_n(\lambda, \tilde\lambda, \eta)}$$

momentum and "super-momentum" conservation.

N.B. for Grassmann number θ: $\quad \delta(\theta) = \theta$.

$\Rightarrow \delta^{(8)}(q)$ factor arises from SUSY invariance of $\mathbb{A}_n$: $\boxed{q^{\alpha A}\mathbb{A}_n = 0}$

$\delta^{(8)}(q) \sim (\eta)^8 \Rightarrow \mathbb{A}_n$ scales at least as η^8

$\Rightarrow$ Certain component amplitudes vanish, in particular

$A_n^{gluon}(+,\ldots,+) = 0$ and $A_n^{gluon}(-,+,\ldots,+)$

(η^0) (η^4)

- Tree level gluon amps in $N=4$ SYM and QCD are identical
  $\rightarrow$ Result carries over "Secret SUSY of massless QCD tree amplitudes".

- The factor $P_n(\lambda,\tilde{\lambda},\eta)$ has expansion in η's. By $SU(4)_R$ symmetry the η^A_i appear as multiplets of 4:

$$\boxed{\begin{array}{ccccccccc} P_n(\lambda,\tilde{\lambda},\eta) = & P_n^{(0)} & + & P_n^{(4)} & + & P_n^{(8)} & + \ldots + & P_n^{(4n-16)} \\ & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ & \text{MHV} & & \text{NMHV} & & \text{N}^2\text{MHV} & & \overline{\text{MHV}} \end{array}}$$

(4n-16) why?

Has (n-4) g- factors.

In particular $P_n^{(0)} = 1$

II.3 SUPER BCFW - RECURSION

It should be possible to construct a super version of BCFW, i.e. augment $\lambda_1 \to \lambda_1 - z\lambda_n$, $\tilde{\lambda}_n \to \tilde{\lambda}_n + z\tilde{\lambda}_1$ by a shift in η. But which?

- <u>Supermomentum conservation</u>: $q^{\alpha A} = \sum_i \lambda_i^\alpha \eta_i^A$

$$\hat{q}_1^{\alpha A} \to (\lambda_1 - z\lambda_n)\hat{\eta}_1 \qquad \text{natural } \eta\text{-shift:}$$

$$q_n^{\alpha A} \to \lambda_n \hat{\eta}_n \qquad \hat{\eta}_n = \eta_n + z\eta_1$$

$$\hat{\eta}_1 = \eta_1$$

Preserves total q: $\hat{q}_1 + \hat{q}_n = q_1 + q_n$

- Also natural w.r.t. $\bar{q}^{\dot\alpha}_A$ symmetry: $\bar{q}^{\dot\alpha}_A = \sum_i \tilde{\lambda}_i^{\dot\alpha} \frac{\partial}{\partial \eta_i^A}$

$\Leftrightarrow$ $\tilde{\lambda}_i$ and η_i are shifted in similar fashion.

$$\Rightarrow \boxed{\begin{aligned} \hat{\lambda}_1 &= \lambda_1 - z\lambda_n \\ \hat{\tilde{\lambda}}_n &= \tilde{\lambda}_n + z\tilde{\lambda}_1 \\ \hat{\eta}_n &= \eta_n + z\eta_1 \end{aligned}}$$

[Brandhuber et al. '08
Arkani-Hamed et al]

- With these shifts one can derive a <u>superamplitude on-shell recursion</u>: (no contribution from $z=\infty$ for any leg):

(*)

$$\boxed{\mathbb{A}_n = \sum_i \int \frac{d^4\eta_{\hat{P}_i}}{P_i^2} \mathbb{A}_L(\hat{1}(z_{P_i}), 2, \ldots, i-1, -\hat{P}(z_{P_i}))\, \mathbb{A}_R(\hat{P}(z_{P_i}), i, \ldots, n-1, \hat{n}(z_{P_i}))}$$

Sum over intermediate states is replaced by Grassmann-integral.

- Building blocks: <u>3-Point Superamplitudes</u>

$$\boxed{\mathbb{A}_3^{MHV} = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}}$$

$$\boxed{\mathbb{A}_3^{\overline{MHV}} = \frac{\delta^{(4)}(p)\ \delta^{(4)}(\eta_1[23]+\eta_2[31]+\eta_3[12])}{[12][23][31]}} \qquad (+)$$

Why?

- Parity $\overline{A_3^{MHV}} = \dfrac{\delta^{(4)}(p)}{[12][23][31]}\ \delta^{(8)}\Big(\sum_{i=1}^{3}\tilde{\lambda}_i\bar{\eta}_i\Big)$

  Grassmann - Fourier - Transformation $\Phi(\eta) = \int d^4\bar{\eta}\ e^{i\eta\bar{\eta}}\ \bar{\Phi}(\bar{\eta})$

  yields $(+)$

- Reason that no $\delta^{(8)}(q)$ appears is that for $\overline{MHV}_3$

  we have $\lambda_1 \propto \lambda_2 \propto \lambda_3 \Rightarrow$ Factorization of $q^{\alpha A} = \lambda_P^{\alpha}\cdot q_F^{A}$

  $\Rightarrow$ q-supersymmetry only requires factor $\delta^{(4)}(q_F^A)$.

DECOMPOSITION OF SUPER-RECURSION:

As RHS of $(*)$ contains $\int d^4\eta$ the sum of η-degrees of $\mathbb{A}_L$ and $\mathbb{A}_R$ has to be 4-times larger than $\mathbb{A}_n$:

$$\boxed{\begin{aligned}\mathbb{A}_n^{N^pMHV} &= \int \frac{d^4\eta_P}{P^2}\ \mathbb{A}_3^{\overline{MHV}}(z_P)\ \mathbb{A}_{n-1}^{N^pMHV}(z_P) \\ &+ \sum_{m=0}^{p-1}\sum_{i=4}^{n-1}\int\frac{d^4\eta_{P_i}}{P_i^2}\ \mathbb{A}_i^{N^mMHV}(z_{P_i})\ \mathbb{A}_{n-i+2}^{N^{(p-m-1)}}(z_{P_i})\end{aligned}}$$

Remarkably this recursion can be solved exactly (Drummond, Henn '08). Let us just quote the NMHV result:

$$\mathbb{A}_n^{NMHV} = \mathbb{A}_n^{MHV}\, P_n^{NMHV} = \mathbb{A}_n^{MHV} \sum_{2\leq a<b\leq n-1} R_{n,ab}$$

R-invariant

$$R_{n,ab} = \frac{\langle a,a-1\rangle\langle b,b-1\rangle\, \delta^{(4)}\big(\langle n|x_{na}x_{ab}|\theta_{bn}\rangle + \langle n|x_{nb}x_{ba}|\theta_{an}\rangle\big)}{x_{ab}^2\,\langle n|x_{na}x_{ab}|b\rangle\langle n|x_{na}x_{ab}|b-1\rangle\langle n|x_{nb}x_{ba}|a\rangle\langle n|x_{nb}x_{ba}|a-1\rangle}$$

$\Downarrow$

Includes all tree-level amplitudes in pure Yang-Mills and single flavour massless QCD. Multi-flavour QCD also possible for up to 8 external quarks (→ Dixon, Henn, JP, Schuster).

$\Downarrow =$ with $\quad x_{ab} := p_i + p_{i+1} + \dots + p_{j-1} = x_a - x_b$

$$\theta_{ij} := q_i + q_{i+1} + \dots + q_{j-1} = \theta_i - \theta_j$$

---

x_i & θ_i have a life of their own: "DUAL COORDINATES":

$$\boxed{\begin{aligned} x_i^{\alpha\dot\alpha} - x_{i+1}^{\alpha\dot\alpha} &= \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha} = p_i^{\alpha\dot\alpha} \, . \\ \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A} &= \lambda_i^\alpha \eta_i^A = q_i^{\alpha\dot\alpha} \end{aligned}}$$

$$x_{ab} = x_a - x_b \qquad \theta_{ab} = \theta_a - \theta_b$$

II.4 DUAL SUPERCONFORMAL SYMMETRY

Surprisingly $\mathcal{A}_n$ is not only invariant und ordinary superconformal symmetries $\{p, m, \bar{m}, q, \bar{q}, r, d, s, \bar{s}, r\}$ but also under a DUAL SUPERCONFORMAL GROUP $\{P, K, M, \bar{M}, Q, \bar{Q}, S, \bar{S}, D, R\}$.

Sketch:

- Building blocks of R-invariants: $x_{ab}^{\alpha\dot\alpha} = x_a^{\alpha\dot\alpha} - x_b^{\alpha\dot\alpha}$ and $\theta_{ab}^{\alpha A} = \theta_a^{\alpha A} - \theta_b^{\alpha A}$ are super-translational invariant

$$P_{\alpha\dot\alpha} = \sum_i \frac{\partial}{\partial x_i^{\alpha\dot\alpha}} \qquad Q_{\alpha A} = \sum_i \frac{\partial}{\partial \theta_i^{\alpha A}}$$

- One shows that building blocks $\langle i\, i{+}1\rangle$, $\langle n|x_{na}x_{ab}|b\rangle$, $\langle n|x_{nb}x_{ba}|\theta_a\rangle$ and x_{ab}^2 transform covariantly under dual inversions: $x^\mu \to -\frac{x^\mu}{x^2}$

$$x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}$$

$$\langle i\, i{+}1\rangle \to \frac{\langle i\, i{+}1\rangle}{x_i^2}$$

$$\langle n|x_{na}x_{ab}|b\rangle \to \frac{\langle n|x_{na}x_{ab}|b\rangle}{x_n^2\, x_a^2\, x_b^2}$$

$$\langle n|x_{na}x_{ab}|\theta_b\rangle \to \frac{\langle n|x_{na}x_{ab}|\theta_b\rangle}{x_n^2\, x_a^2\, x_b^2}$$

As $K_\mu = I \circ P_\mu \circ I$ one has covariance of $\mathbb{A}_n$ under special dual conformal transformations:

$$K_{\alpha\dot\alpha} = \sum_i x_{i\alpha}{}^{\dot\beta}\, x_{i\dot\alpha}{}^{\beta}\, \partial_{i\beta\dot\beta}$$

with $K_{\alpha\dot\alpha}\, \mathbb{A}_n = - \sum_i x_i^{\alpha\dot\alpha}\, \mathbb{A}_n$. (Drummond, Henn, Korchemsky, Sokatchev, '08)

Similarly: $S^{\alpha A}\, \mathbb{A}_n = - \sum_i \theta_i^{\alpha A}\, \mathbb{A}_n$ dual super-conformal

$D\, \mathbb{A}_n = n\, \mathbb{A}_n$ dual dilatation

with $S^{\alpha A} = \sum_i - \theta_i^{\alpha B}\, \theta_i^{\beta A}\, \partial_{i\beta B} + x_{i}{}^{\alpha\dot\beta}\, \theta_i^{\beta A}\, \partial_{i\beta\dot\beta}$

$$D = \sum_i - x_i^{\alpha\dot\alpha}\, \partial_{i\alpha\dot\alpha} - \tfrac{1}{2}\, \theta_i^{\alpha A}\, \partial_{i\alpha A}$$

Hence the redefined generators $K' = K + \sum_i x_i$, $S' = S + \sum_i \theta_i$, $D' = D - n$ annihilate the tree-level superamplitudes.

Summary: superconformal & dual superconformal symmetry:

$$\{k, d, s, \bar s;\ p, m, \bar m, r\} \circ \mathbb{A}_n = 0$$

$$\{K', D', S', \bar S;\ P, M, \bar M, R\} \circ \mathbb{A}_n = 0$$

But two sets of generators act in different but related spaces:

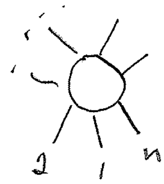
Constraints: $x_i^{\alpha\dot\alpha} - x_{i+1}^{\alpha\dot\alpha} = \lambda_i^{\alpha}\, \tilde\lambda_{i+1}^{\dot\alpha}$ $\quad \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A} = \lambda_i^{\alpha}\, \eta_i^{A}$

II.5 YANGIAN SYMMETRY OF TREE-LEVEL SUPERAMPLITUDE

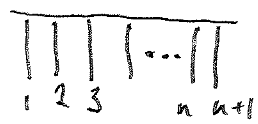
Q: What algebra emerges when we commute conf. and dual conformal generators? (Henn, Drummond, JP).

Task: Needs to express (x_i, θ_i) in terms of $(\lambda_i, \tilde{\lambda}_i, \eta_i)$

① Open the chain: by introducing: x_{n+1} & θ_{n+1}



$\rightarrow$



$$\delta^{(4)}(p)\, \delta^{(8)}(q) = \delta^{(4)}(x_1 - x_{n+1})\, \delta^{(8)}(\theta_1 - \theta_{n+1})$$

δ-Functions impose closed chain.

② Express dual variables via non-local relations:

$$x_i = x_1 + \sum_{j<i} \lambda_j \tilde{\lambda}_j \qquad \theta_i = \theta_1 + \sum_{j<i} \lambda_j \eta_j$$

Use dual supertranslations (P, Q) to set $x_1 = \theta_1 = 0$

③ Reexpress dual conformal generators as differential operators acting in original on-shell superspace $\{\lambda_i, \tilde{\lambda}_i, \eta_i\}$

$\Rightarrow$ Only $K'_{\alpha\dot{\alpha}}$ and $S'^{\alpha A}$ remain as novel structures, all other related to $\{r, d, s, \bar{s}, p, q, \bar{q}, m, \bar{m}, r\}$.

E.g.

$$K^{\alpha\dot\alpha} = \sum_{i=1}^{n} \underbrace{x_i^{\dot\alpha\beta}}_{=\sum_{j=1}^{i-1}\lambda_j^{\dot\alpha}\lambda_j^{\beta}} \lambda_i^{\alpha} \frac{\partial}{\partial\lambda_i^{\beta}} + \underbrace{x_{i+1}^{\alpha\dot\beta}}_{=\sum_{j=1}^{i}\lambda_j^{\alpha}\lambda_j^{\dot\beta}} \tilde\lambda_i^{\dot\alpha} \frac{\partial}{\partial\tilde\lambda_i^{\dot\beta}} + \text{ferm}$$

Non-local structure!

In general finds Yangian algebra $\underline{Y[psu(2,2|4)]}$:

Let $J_a^{(0)} = \sum_i J_{i,a}^{(0)}$ global $psu(2,2|4)$ generators

$$\boxed{[J_a^{(0)}, J_b^{(0)}] = f_{ab}{}^{c} J_c^{(0)}}$$

Non-local level one gen: $J_a^{(1)} = f^{cb}{}_a \sum_{1\le j<i\le n} J_{i,b}^{(0)} J_{j,c}^{(0)}$

$$\boxed{[J_a^{(1)}, J_b^{(0)}] = f_{ab}{}^{c} J_c^{(1)}}$$

∞-dim symmetry algebra of tree-level amps!

$$\boxed{y \circ A_n = 0} \quad \text{with} \quad y \in Y[psu(2,2|4)]$$

Simple form of generators in super twistor space variables Z^{A}:

F.T $\lambda \to \tilde{\mu}$: $\quad Z^{A} = (\tilde{\mu}^{\alpha}, \tilde{\lambda}^{\dot\alpha}, \eta^{A})$

$$\Rightarrow \boxed{\begin{aligned} (J^{(0)})^{A}{}_{B} &= \sum_{i} Z_i^{A} \frac{\partial}{\partial Z_i^{B}} \\ (J^{(1)})^{A}{}_{B} &= \sum_{i<j} (-)^{|C|} \left[Z_i^{A} \frac{\partial}{\partial Z_i^{C}} Z_j^{C} \frac{\partial}{\partial Z_j^{B}} - i \leftrightarrow j \right] \end{aligned}}$$

<u>Challenge</u>: Extend the symmetry to loop level!