# 11 Quantum Theory of Maser and Laser

One example for a system interacting with reservoirs is the maser or laser.

In order to understand the properties of such an inherently quantum device a quantum theory is required for the theoretical description, in particular if maser action or lasing with only a single quantum emitter occurs.

The following figure sketches the various subcomponents (systems and reservoirs) of a laser:

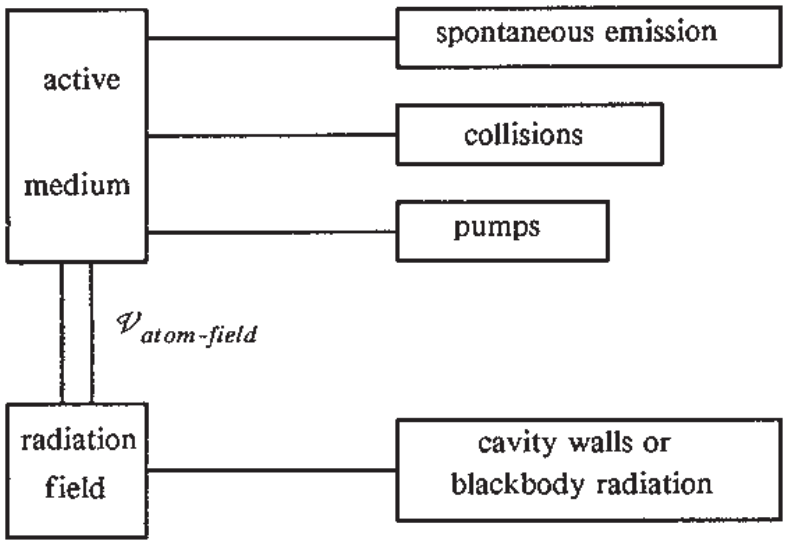


Figure 76: Block diagram of the subcomponents of a laser [from Meystre "Elements of Quantum optics"]

## 11.1 The Micromaser

The most fundamental system to study light-matter interaction and light amplification is the one-atom maser or micromaser.

In this system single atoms from an atomic beam are sent through a high quality cavity.

Starting point for a theoretical description is the Jaynes-Cummings-Hamiltonian:

$$\mathcal{H} = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega a^\dagger a + \hbar(g\sigma_+ a + g^*\sigma_- a^\dagger) \tag{545}$$

The time evolution of the atom density matrix is subdivided into different steps (Filipowicz et al. Phys. Rev. A 34, 3077 (1986)).

First, an atom enters the cavity at time $t_i$ and interacts for a time $\tau$. Then the density matrix of the atom $\rho$ can be found by tracing the total density matrix $\rho_{a-f}$ over the field variables after interaction due to the time evolution operator $U$:

$$\rho(t_i + \tau) = tr_{atom}\{U(\tau)\rho_{a-f}(t_i)U^{-1}(\tau)\} \equiv F(\tau)\rho(t_i) \tag{546}$$

After that the density matrix of the field evolves according to a master equation:

$$\dot{\rho} \equiv L\rho = -\frac{\omega}{2Q}(\overline{n_R} + 1)[a^\dagger a\rho(t) - a\rho(t)a^\dagger] - \frac{\omega}{2Q}\overline{n_R}[\rho(t)aa^\dagger - a^+\rho(t)a] + adj. \tag{547}$$

The Q-factor $Q$ is defined as the ratio of the resonance frequency and the resonance width $Q = f/\Delta f$ and is thus inversely proportional to the damping rate of the cavity.

Thus, after the coherent and incoherent interaction the density matrix of the field is:

$$\rho(t_{i+1}) = exp(Lt_p)F(t)\rho(t_i) \tag{548}$$

Then, the next atom enters. Damping of the field is neglected during the atom field interaction!

One special case is when the field is initially diagonal in the energy representation and when the atom is injected in its upper state $|e\rangle$. Then the reduced density operator of the field remains diagonal.

With the initial density matrix

$$\rho_{a-f}(t_i) = |e\rangle\langle e| \otimes \sum_n p_n(t_i)|n\rangle\langle n| \tag{549}$$

one finds:

$$\rho(t_i + \tau) = \sum_n p_n(t_i)[|C_{en}(\tau)|^2|n\rangle\langle n| + |C_{gn+1}(\tau)|^2|n+1\rangle\langle n+1|] \tag{550}$$

and

$$p_n(t_i + \tau) = [1 - \mathcal{C}_{n+1}(\tau)]p_n(t_i) + \mathcal{C}_n(\tau)p_{n-1}t(i) \tag{551}$$

Where the $C_n$ are:

$$\mathcal{C}_n(\tau) = \frac{4ng^2}{(\omega_0 - \omega)^2 + 4ng^2} \sin^2\left[\frac{1}{2}\sqrt{(\omega_0 - \omega)^2 + 4ng^2}\tau\right] \tag{552}$$

Since the density matrix of the field remains diagonal, the master equation can be limited to the diagonal elements:

$$\dot{p}_n = -\frac{\omega}{Q}(\overline{n_R} + 1)[np_n - (n+1)p_{n+1}] - \frac{\omega}{Q}\overline{n_R}[(n+1)p_n - np_{n-1}] \tag{553}$$

Iteration of the time evolution eventually yields to a steady state.

One can calculate the steady state properties if one assumes that the atoms enter in a Poissonian distribution at a rate $R$ and an average distance in time of $1/R$. Averaging yields:

$$\langle\rho(t_{i+1})\rangle = \langle exp(Lt_p)\rangle F(\tau)\langle\rho(t_i)\rangle = \tag{554}$$
$$= R\int_0^\infty dt_p exp[-(R-L)t_p]F(\tau)\langle\rho(t_i)\rangle \tag{555}$$
$$= \frac{R}{R-L}F(\tau)\langle\rho(t_i)\rangle \tag{556}$$

In steady state it is:

$$R[1-F(\tau)]\overline{\rho} = L\overline{\rho} \tag{557}$$

Now the diagonal elements $p_n = \langle n|\rho|n\rangle$ can be found in steady state. A short calculation yields:

$$\overline{p}_n = \frac{\overline{n_R}\omega/Q + \mathcal{A}_n}{(\overline{n_R}+1)\omega/Q + \mathcal{B}_n}\overline{p}_{n-1} \tag{558}$$

where the coefficients $\mathcal{A}_n$ (change of the matrix element $p_n$ per time due to the passage of atoms in the upper state) and $\mathcal{B}_n$ (change of the matrix element $p_n$ per time due to the passage of atoms in the lower state) are:

$$\mathcal{A}_n = \frac{4R_e g^2}{(\omega_0-\omega)^2+4ng^2}\sin^2\left[\frac{1}{2}\sqrt{(\omega_0-\omega)^2+4ng^2}\tau\right] \tag{559}$$
$$\mathcal{B}_n = \frac{4R_g g^2}{(\omega_0-\omega)^2+4ng^2}\sin^2\left[\frac{1}{2}\sqrt{(\omega_0-\omega)^2+4ng^2}\tau\right] \tag{560}$$

Here the rate $R$ has been generalized for the case where atoms are injected in the upper state with rate $R_e$ or in the lower state with rate $R_g$.

Finally, one finds an analytic expression for the steady state photon number distribution in the micromaser with incoherent Poissonian pump:

$$\overline{p}_n = \overline{p}_0\prod_{k=1}^{n}\frac{\overline{n_R}\omega/Q + \mathcal{A}_k}{(\overline{n_R}+1)\omega/Q + \mathcal{B}_k} \tag{561}$$

#### 11.1.1 Features of the photon statistics

With the steady state solution of the photon number distribution the moments $\langle n\rangle$, $\langle n^2\rangle$, ...can be calulated.

A parameter which is often used is the normalized average photon number $n$

$$n \equiv \langle n\rangle/N_e = \sum_n n\overline{p}_n/N_e \tag{562}$$

and the dimensionless pump parameter $\Theta$ :

$$\Theta = \frac{1}{2}\sqrt{N_e}g\tau \tag{563}$$

The following plot shows the mean photon number as a function of the pump parameter for three different atom injection rates:

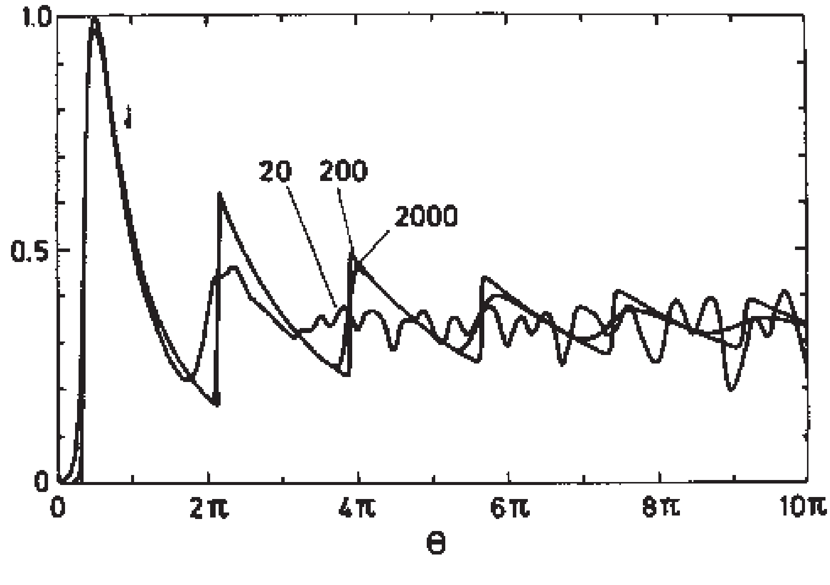


Figure 77: Normalized average photon number for different values of $R/\gamma$ [Filipowicz et al. Phys. Rev. A 34, 3077 (1986)]

The clearly observable oscillation is a reminiscent of the Rabi-oscillation.

Another interesting property is the non-classical character of the micromaser field in steady state. This is best characterized by the normailzed variance $\sigma$:

$$\sigma \equiv \frac{(\langle n^2 \rangle - \langle n \rangle^2)^{1/2}}{\langle n \rangle^{1/2}} \tag{564}$$

The following figure shows $\sigma$ versus the pump parameter.

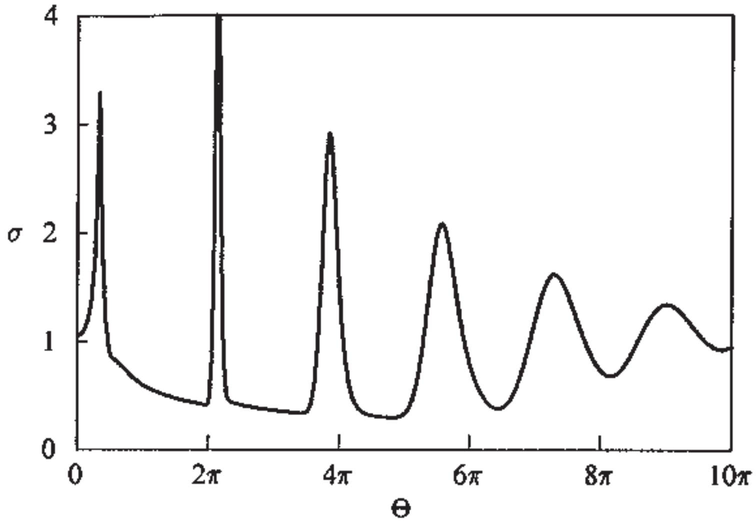


Figure 78: Normailzed variance of the micromaser field in steady state. [from Meystre "Elements of Quantum Optics"]

In regions of increasing photon number a super-Poissonian statistics is observed, whereas in regions of decreasing photon number the field shows a non-classical sub-Poissonian statistics.

At very low temperatures (where thermal photons are negligible) an interesting behaviour of the steady state field occurs: If the photon number in the cavity $n_q$ is such that the atom performs an exact integer multiple $q$ of full Rabi-oscillation, then the probability to emit an additional photon is zero. The photon number is "trapped" at a certain level $n_q$. Only losses are compensated.

The condition for these *trapping states* is thus:

$$g\sqrt{n_q + 1}\tau = 2q\pi \quad \text{or} \tag{565}$$

$$\frac{N_e}{\Theta^2} = \frac{n_q + 1}{q^2\pi^2} \tag{566}$$

Trapping states approximate Fock states under certain conditions.

The following plot shows the normalized mean photon number and variance for a very low number of thermal photons:

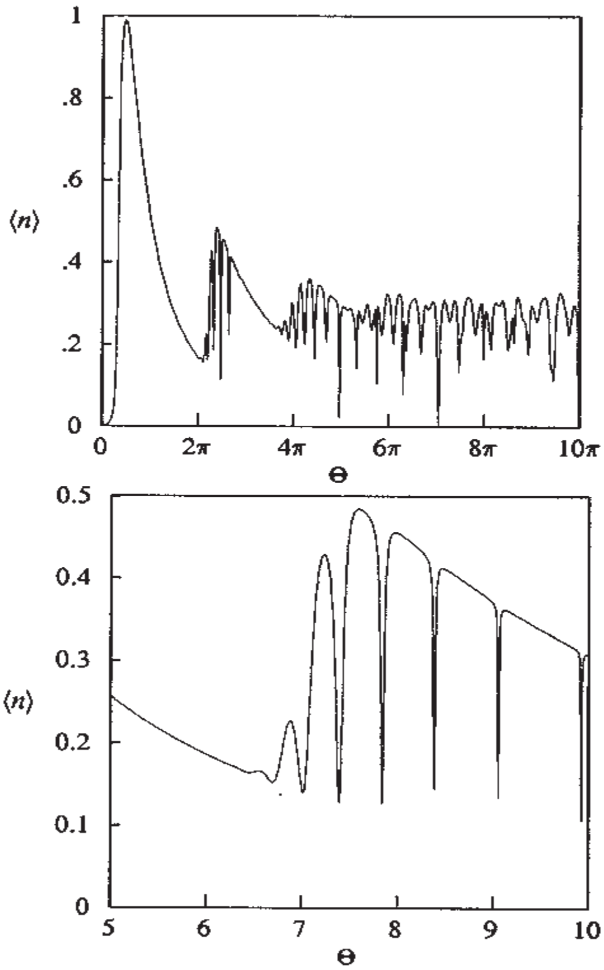


Figure 79: Normalized mean photon number and variance for an average thermal photon number of $10^{-7}$. [from Meystre "Elements of Quantum Optics"]

The next figures show an experiment [Weidinger et al., Phys. Rev. Lett. 82, 3795 (1999)] where trapping states were demonstrated in a one-atom-maser.

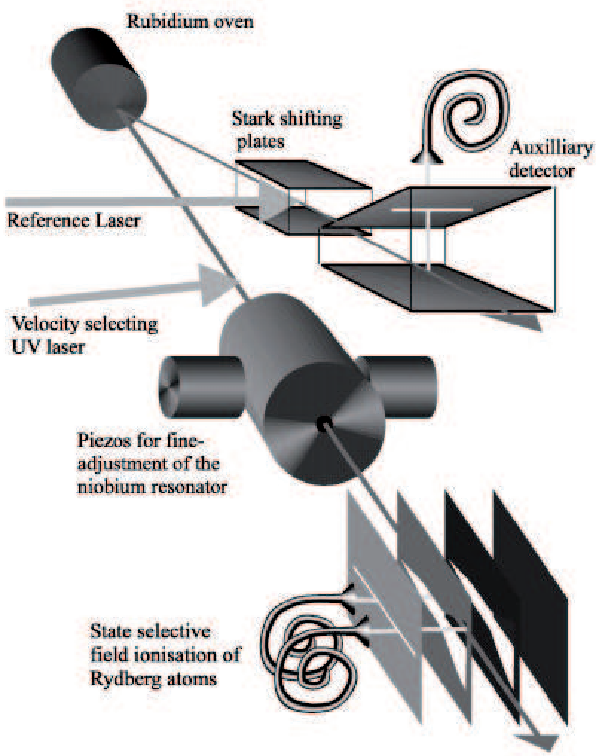


Figure 80: The experimental: Atoms leaving the rubidium oven are excited into the $63P_{3/2}$ Rydberg state using a tilted UV laser for velocity selection. Following the cavity the atoms are detected using state selective field ionization. Tuning of the cavity is performed using two piezo translators. The reference beam is used to stabilize the laser frequency to a Stark shifted atomic resonance, allowing the velocity of the atoms to be tuned continuously. [from Weidinger et al., Phys. Rev. Lett. 82, 3795 (1999)]

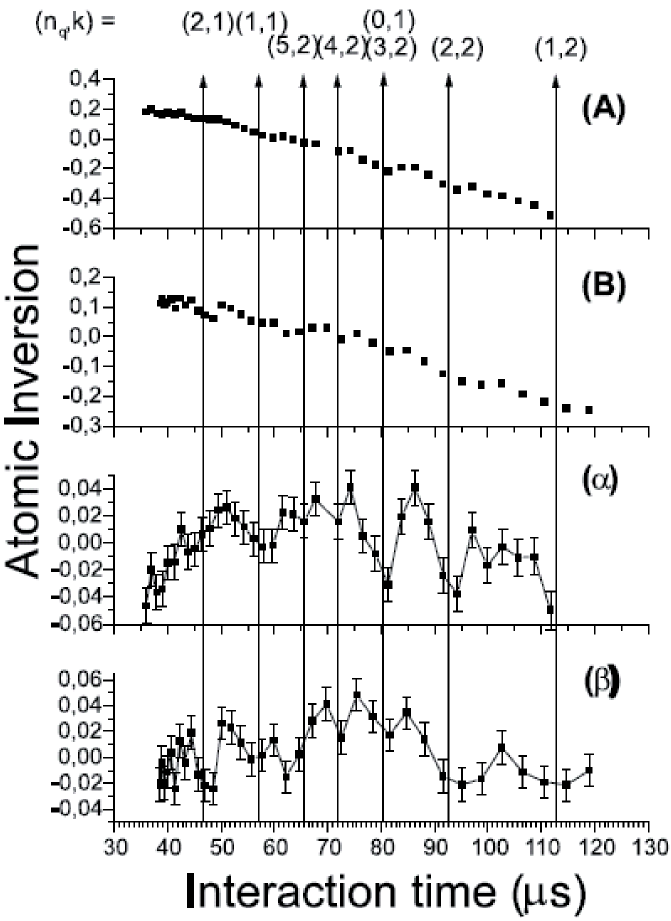


Figure 81: Atomic inversion as a function of interaction time. Plots (A) and (B) present the inversion as a function of interaction time for pump rates of $N_{ex} = 7$ and $N_{ex} = 10$, respectively. Plots (a) and (b) represent the plots (A) and (B), respectively, after the linear trend was removed (see text). The vertical lines on the plot indicate the theoretical positions of all low order trapping states over the range of interaction times of the plot. Dips in the inversion can be identified as corresponding with the positions of the indicated trapping states. [from Weidinger et al., Phys. Rev. Lett. 82, 3795 (1999)]

### 11.2 Single mode laser master equation

The generalization of a single atom maser to laser is to allow that many atoms interact with the cavity field.

In this regime collective effects (superfluorescence, superradiance) may occur if the atoms 'see' the same field. However, here we assume that the atoms move over distances large compared to the optical wavelength and respond independently to the field.

Then it is easy to combine the expressions which were obtained in the previous section to obtain the time evolution for the diagonal elements of the density matrix in the laser.

A so-called coarse-grained derivative can be defined as follows:

$$\dot{p}_n = \frac{p_n(t_i + \tau) - p_n(t_i)}{\tau} \tag{567}$$

If we now combine the terms $\mathcal{A}_n$ (change of the matrix element $p_n$ per time due to the passage of atoms in the upper state) and $\mathcal{B}_n$ (change of the matrix element $p_n$ per time due to the passage of atoms in the lower state) and the losses of the field (described by the master equation) we find:

$$\begin{aligned}\dot{p}_n = &-(n+1)\left[\mathcal{A}_{n+1} + \frac{\omega}{Q}\overline{n}\right] p_n + (n+1)\left[\mathcal{B}_{n+1} + \frac{\omega}{Q}(\overline{n}+1)\right] p_{n+1} \\ &+ n\left[\mathcal{A}_n + \frac{\omega}{Q}\overline{n}\right] p_{n-1} - n\left[\mathcal{B}_n + \frac{\omega}{Q}(\overline{n}+1)\right] p_n\end{aligned} \tag{568}$$

This equation is valid if the matrix element does not change much due to the passage of a single atom and if the cavity losses during the interaction of an atom with the cavity can be neglected.

In order to derive the equation of motion for the maser we averaged the arrival time of the atoms over a Poissonian distribution and assumed a fixed interaction time. Here we assume that the atoms are all present in the cavity, but we average over the lifetimes $T_1$ of the atoms in the state $|e\rangle$ and $|g\rangle$. For simplicity we assume that these are given by the same rate $\gamma$.

Averaging with $\gamma \int d\tau e^{-\gamma\tau}$ yields:

$$\mathcal{A}_n = \frac{R_e R}{2(1+nR)} \tag{569}$$

$$\mathcal{B}_n = \frac{R_g R}{2(1+nR)} \tag{570}$$

where the following dimensionless rate constant $R$ is defined as

$$R = 4|g/\gamma|^2 \mathcal{L}(\omega_0 - \omega) \tag{571}$$

and $\mathcal{L}(\omega_0 - \omega)$ is a dimensionless Lorentzian:

Here, $N_e = R_e T_1$ and $N_g = R_g T_1$ describe the average number of atoms present in the excited and ground state, respectively.

$$\mathcal{L}(\omega_0 - \omega) = \frac{\gamma^2}{\gamma^2 + (\omega_0 - \omega)^2} \tag{572}$$

In the next subchapter we have to solve the equations of motion under these assumption.

### 11.3 Laser Photon Statistics and Linewidth

In order to find a solution for the photon distribution of the laser in steady state it is useful to introduce the following notation:

$$\mathcal{A}'_n = \mathcal{A}_n + \frac{\omega}{Q}\overline{n_R} \text{ and } \mathcal{B}'_n = \mathcal{B}_n + \frac{\omega}{Q}(\overline{n_R} + 1) \tag{573}$$

and with this to write:

$$\dot{p}_n = -(n+1)\mathcal{A}'_{n+1} p_n + (n+1)\mathcal{B}'_{n+1} p_{n+1} + n\mathcal{A}'_n p_{n-1} - n\mathcal{B}'_n p_n \tag{574}$$

This equation can be solved in steady state ($\dot{p}_n = 0$) with the help of the detailed balance condition. With this one can find:

$$p_n = p_0 \prod_{k=1}^{n} \frac{\overline{n_R}\omega/Q + \mathcal{A}_k}{(\overline{n_R}+1)\omega/Q + \mathcal{B}_k} \tag{575}$$

For an optical field one can set the mean number of thermal photons $\overline{n_R}$ to zero. By inserting the expressions for $\mathcal{A}'_n$ and $\mathcal{B}'_n$ one finds:

$$p_n = p_0 \prod_{k=1}^{n} \frac{N_a}{N_b + 2T_1(R^{-1}+k)\omega/Q} \tag{576}$$

The following figure shows the steady state photon number distribution according to this photon number distribution:

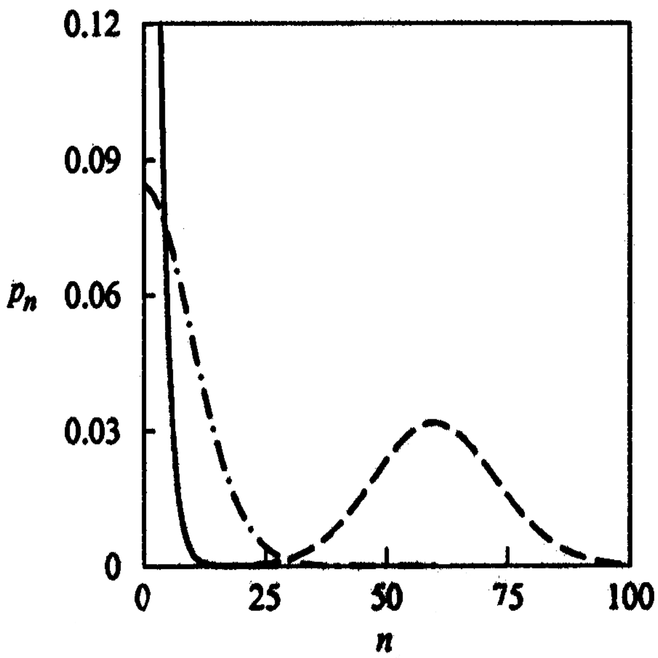


Figure 82: Photon number distribution of a laser in steady state below (solid line) and above (dashed line) threshold [from Meystre "Elements of Quantum Optics"]

Far below threshold the photon number distribution has the form:

$$p_n = p_0 x^n = (1-x)x^n \quad \text{with} \tag{577}$$
$$x = (N_e/(N_g + 2T_1\omega/QR)) \tag{578}$$

and thus resembles a thermal distribution.

One can write the expression for the $p_n$ as:

$$p_n = \frac{P_0}{(n+C)!}\left[\frac{N_e}{2T_1(\omega/Q)}\right]^{n+C} \quad \text{with} \tag{579}$$
$$C = R^{-1} + N_g/2T_1(\omega/Q) \tag{580}$$

The average photon number in steady state is thus:

$$n_{ss} = \sum_n np_n = p_0 \sum_n \frac{n+C-C}{(n+C)!}\left[\frac{N_e}{2T_1(\nu/Q)}\right]^{n+C} \tag{581}$$
$$= \frac{N_e - N_g}{2T_1\nu/Q} - \frac{1}{R} \tag{582}$$

Far above threshold the $n$'s around the average value $< n >$ are much larger than $C$. Then the following approximation holds:

$$p_n = \frac{e^{-\langle n\rangle}\langle n\rangle^n}{n!} \tag{583}$$

*The photon number distribution of a laser far above threshold is thus Poissonian. The state of the laser field is a "phase diffused" coherent state, i.e. not a pure state.*

The average value of the photon number evolves as:

$$
\begin{aligned}
\frac{d}{dt}\langle n\rangle &= \sum_n n\dot{p}_n = -\sum_n [\mathcal{A}'_{n+1}(n^2+n) - \mathtt{B}'_n(n^2-n) \\
&- \mathcal{A}'_{n+1}(n^2+2n+1) + \mathcal{B}'_n n^2]p_n \\
&= \sum_n (\mathcal{A}'_{n+1} - \mathcal{B}'_n) n p_n + \sum_n \mathcal{A}'_{n+1} p_n \\
&\simeq (\mathcal{A} - \mathcal{B} - \omega/Q)\langle n\rangle + \mathcal{A} + \overline{n_R}\omega/Q
\end{aligned}
\tag{584}
$$

In the last row the values $\mathcal{A}'_{n+1}$ and $\mathcal{B}'_n$ were replaced by their semiclassical values:

$$\mathcal{A} = \frac{RN_e}{2T_1(1+\langle n\rangle)R} \tag{585}$$

$$\mathcal{B} = \frac{RN_g}{2T_1(1+\langle n\rangle)R} \tag{586}$$

This is appropriate if the photon number distribution is strongly peaked around its average value.

The photon number distribution thus builds up (even if there are initially no photons, $\langle n\rangle = 0$ !) from the thermal photons or from vacuum fluctuations.

The last term $\mathcal{A} + \overline{n}\nu/Q$ describes the time evolution of the photon number fluctuations and gives rise to a finite laser linewidth:

$$\Delta\nu = \frac{\mathcal{A} + \overline{n_R}\omega/Q}{n_{ss}} \tag{587}$$