QUANTUM OPTICS
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Zusatzübungsblatt
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Density matrix formulation

For a given physical system, there exists a state vector $|\psi\rangle$ which contains all possible information about the system. If we want to extract a piece of the system's information, we must calculate the expectation value of a corresponding operator $\hat{O}$:

$$\left\langle \hat{O} \right\rangle_{\mathrm{QM}} = \langle\psi|\hat{O}|\psi\rangle$$

In many situations we may not know $|\psi\rangle$, we may only know the probability P_ψ that the system is in the state $|\psi\rangle$. For such a situation, we not only need to take the quantum mechanical average, but also the ensemble average over many identical systems that have been similarly prepared:

$$\left\langle \left\langle \hat{O} \right\rangle_{\mathrm{QM}} \right\rangle_{\mathrm{ensemble}} = \mathrm{Tr}(\hat{O}\hat{\rho}) = \sum_b \langle b|\hat{O}\hat{\rho}|b\rangle$$

Here, $\{|b\rangle\}$ is any complete ortho-normal basis of the corresponding Hilbert space. The density matrix $\hat{\rho}$ is defined by:

$$\hat{\rho} = \sum_\psi P_\psi |\psi\rangle\langle\psi|$$

In the particular case where all P_ψ are zero except for the one for a state $|\psi_0\rangle$, then

$$\hat{\rho} = |\psi_0\rangle\langle\psi_0|,$$

and the state is called a pure state.
The most general form of the density matrix in a basis $\{|b\rangle\}$ is

$$\hat{\rho} = \sum_{i,j} \rho_{i,j} |b_i\rangle\langle b_j|.$$

The coefficients $\rho_{i,j}$ are in general complex, but the diagonal elements $\rho_{i,i}$ are positive and real.

(a) Show that the trace is invariant under permutation, i.e. $\mathrm{Tr}(\hat{A}\hat{B}\hat{C}) = \mathrm{Tr}(\hat{B}\hat{C}\hat{A}) = \mathrm{Tr}(\hat{C}\hat{A}\hat{B})$.

(b) Show that (I.) $\mathrm{Tr}(\hat{\rho}) = 1$, (II.) that $\mathrm{Tr}(\hat{\rho}^2) \leq 1$, and (III.) that $\mathrm{Tr}(\hat{\rho}^2) = 1$ if and only if $\hat{\rho}$ describes a pure state.

(c) Write the density matrix for the entangled state $|\psi\rangle = 2^{-1/2}(|H,V\rangle + |V,H\rangle)$. Compare it with the density matrix of a classically correlated state $\hat{\rho}_{\mathrm{class}} = \frac{1}{2}|H,V\rangle\langle H,V| + \frac{1}{2}|V,H\rangle\langle V,H|$.

(d) Take the density matrix from the entangled state in (c). Show that when detecting photon 1 while ignoring its state (e.g. a polarization-insensitive photo-detector), the state of photon 2 becomes a classically mixed state. Hint: a measurement on one particle only is described by tracing over the corresponding subspace, i.e. $\mathrm{Tr}_1\hat{\rho} = \sum_a \langle a,\cdot|\rho|a,\cdot\rangle$.

(e) The time evolution of a pure state $|\psi\rangle$ is given by the Schrödinger equation:

$$\frac{d}{dt}|\psi\rangle = -\frac{i}{\hbar}\mathcal{H}|\psi\rangle.$$

Show that the corresponding equation of motion for the density matrix is

$$\frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar}[\mathcal{H}, \hat{\rho}].$$