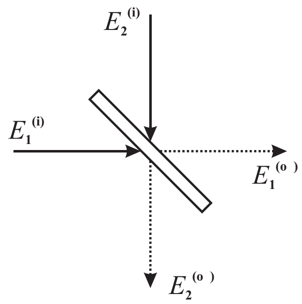
QUANTUM OPTICS
Wintersemester 2008/2009

Blatt 3
zur Übung am 3. November 2008

Exercise 1: The beamsplitter



A classical beamsplitter can be described as a 2×2 matrix that relates the input fields $E_1^{(i)}$, $E_2^{(i)}$ to the output fields $E_1^{(o)}$, $E_2^{(o)}$.

(a) Find such a matrix M using the parameter $\sqrt{R}$ where R is the reflection coefficient (for the intensity). Be careful to use the right phase shift (reflection on an optically thicker interface).

(b) What is the intensity in the output arms for arbitrary classical input fields?

In the quantum case the same matrix relates the operators $\hat{a}_1^{(i)}$, $\hat{a}_2^{(i)}$ and $\hat{a}_1^{(o)}$, $\hat{a}_2^{(o)}$ and their Hermitian conjugates.

(c) What is the intensity in the outputs 1 and 2 if a single photon enters arm 1 (i.e. the initial state is $|1\rangle_1^{(i)} = \hat{a}_1^{\dagger(i)}|0\rangle$) or arm 2 (i.e. the initial state is $|1\rangle_2^{(i)} = \hat{a}_2^{\dagger(i)}|0\rangle$)?

(d) Show that a coherent state, when entering an input port, is split in a product of two coherent states in the output ports.

(d) What happens if two identical photons enter arm 1 and arm 2 at the same time (i.e. the initial state is $\hat{a}_1^{\dagger(i)}\hat{a}_2^{\dagger(i)}|0\rangle$)? Consider the special case of a 50:50 beamsplitter ($R = 0.5$).

(e) We can introduce the polarization as a further degree of freedom. In this case, the operators $\hat{a}_{x,H}^{\dagger(i)}$ and $\hat{a}_{x,V}^{\dagger(i)}$ create a photon with horizontal (H) and vertical (V) polarization, respectively, in arm x ($x = 1, 2$). How do the results in (d) change when the two photons in either arm have the same / opposite polarization?

(f) Generalize the result from (e) to arbitrary degrees of freedom. How can this type of interference experiment be used to "compare" two individual photons?

Exercise 2: $G^{(2)}$ function for quantum fields

Consider the electric field operator

$$E^+(\mathbf{r}_i,t) = E_0\big(\hat{a}_{\mathbf{k}} e^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{r}_i)} + \hat{a}_{\mathbf{k}'} e^{-i(\omega_{k'} t - \mathbf{k}'\cdot\mathbf{r}_i)}\big)$$

from two quantum sources S and S', assuming equal frequencies ($\omega = c|\mathbf{k}| = c|\mathbf{k}'|$). The second order correlation function $G^{(2)} = \langle E^-(\mathbf{r}_1,t)E^-(\mathbf{r}_2,t)E^+(\mathbf{r}_2,t)E^+(\mathbf{r}_1,t)\rangle$ is given by

$$\begin{aligned} G^{(2)} &= E_0^4 \langle \hat{a}_{\mathbf{k}}{}^\dagger \hat{a}_{\mathbf{k}}{}^\dagger \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}'}{}^\dagger \hat{a}_{\mathbf{k}'}{}^\dagger \hat{a}_{\mathbf{k}'} \hat{a}_{\mathbf{k}'} \\ &+ \hat{a}_{\mathbf{k}}{}^\dagger \hat{a}_{\mathbf{k}'}{}^\dagger \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}'} \left(1 + e^{-i(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{r}_1-\mathbf{r}_2)}\right) \\ &+ \hat{a}_{\mathbf{k}'}{}^\dagger \hat{a}_{\mathbf{k}}{}^\dagger \hat{a}_{\mathbf{k}'} \hat{a}_{\mathbf{k}} \left(1 + e^{+i(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{r}_1-\mathbf{r}_2)}\right) \end{aligned}$$

Calculate $G^{(2)}$ for

a) thermal light.

b) laser (poissonian) light.

c) Fock states.

Express the results in terms of the average photon number $\langle n \rangle$.

Exercise 3: Wigner function

The Wigner function $W(q,p)$ is the quasi-probability distribution of the conjugate variables q and p. The Wigner functions for coherent states and Fock states are given by

$$\begin{aligned} W_{coh}(Q,P) &= \frac{2}{\pi} e^{-\frac{1}{2}(Q^2+P^2)} \quad \text{and} \\ W_{Fock}(Q,P) &= \frac{2}{\pi} (-1)^n L_n(4(Q^2+P^2)) e^{-2(Q^2+P^2)}, \end{aligned}$$

respectively, with $Q \propto q$, $P \propto p$ and $L_n(x)$ being the Laguerre polynomial.

Plot $W(Q,P)$ for a coherent state and a single-photon Fock state. Where do these functions show non-classical behaviour? How can we interpret these plots with respect to mean electric field $\langle E \rangle$, mean intensity $\langle |E|^2 \rangle$, and phase?