QUANTUM OPTICS
Wintersemester 2008/2009

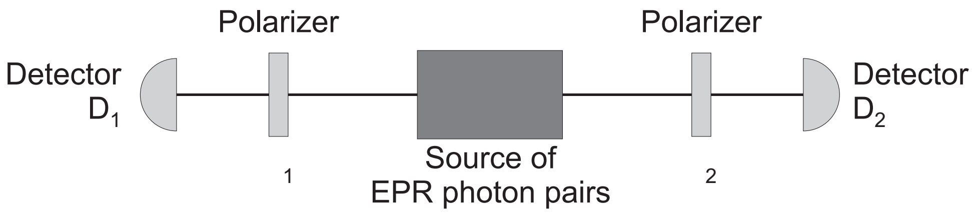
Blatt 4
zur Übung am 24. November 2008

Exercise 1: Bell's inequality

An entangled state produced by the source of photon pairs has the form:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\big(|1_{1x}, 0_{1y}, 0_{2x}, 1_{2y}\rangle - |0_{1x}, 1_{1y}, 1_{2x}, 0_{2y}\rangle\big),$$

which means that it is either one photon in arm 1 with x-polarization and one in arm 2 with y-polarization or vice versa.



The relation between the annihilation (creation) operators after the polarizer and the operators before the polarizer is:

$$a_j = a_{jx}\cos\theta_j + a_{jy}\sin\theta_j, \qquad j = 1, 2$$

a) What are the probabilities $P_1(\theta_1) = \langle\Psi|a_1^+ a_1|\Psi\rangle$, $P_2(\theta_2) = \langle\Psi|a_2^+ a_2|\Psi\rangle$ to detect a photon after the polarizer 1 and 2, respectively, when the polarizers are set to θ_1, θ_2?

b) What is the joint probability $P_{12}(\theta_1, \theta_2) = \langle\Psi|a_1^+ a_2^+ a_2 a_1|\Psi\rangle$ that one photon is detected after polarizer 1 at angle θ_1 and one after polarizer 2 at angle θ_2?

c) What is the conditional probability $P_{\theta_1}(\theta_2) = P_{12}(\theta_1, \theta_2)/P_1(\theta_1)$ to detect photon 2 in arm 2, given the detection of photon 1 in arm 1? Has the outcome of the measurement in arm 1 any influence on the measurement in arm 2?

d) Calculate the joint probabilities $P(+, \theta_1, +, \theta_2)$ that one photon emerges from polarizer 1 and one photon emerges from polarizer 2, $P(-, \theta_1, -, \theta_2)$ that no photon emerges from either arm and the probabilities $P(+, \theta_1, -, \theta_2)$, $P(-, \theta_1, +, \theta_2)$ that one photon emerges from polarizer 1 and no photon from polarizer 2 and vice versa.

e) In Bell's inequality one considers the correlation

$$C(\theta_1, \theta_2) = P(+, \theta_1, +, \theta_2) + P(-, \theta_1, -, \theta_2) - P(+, \theta_1, -, \theta_2) - P(-, \theta_1, +, \theta_2)$$

and defines the parameter S as

$$S = |C(\theta_1, \theta_2) - C(\theta_1, \theta_2')| + |C(\theta_1', \theta_2) + C(\theta_1', \theta_2')|.$$

Calculate S for the entangled state given above and the polarizer settings $\theta_1 = 0$, $\theta_2 = 3\pi/8$, $\theta_1' = -\pi/4$, and $\theta_2' = \pi/8$.

f) The classical correlation is given by

$$C(\theta_1, \theta_2) = \langle A(\theta_1) B(\theta_2) \rangle = \int A(\theta_1, \lambda) B(\theta_2, \lambda) g(\lambda) d\lambda$$

where $A(\theta_1, \lambda)$, $B(\theta_2, \lambda) \in \{-1, +1\}$, λ is some unknown parameter and $g(\lambda)$ some classical probability distribution. Show that $S \leq 2$ in the classical case.

(Hint: find inequalities for $|C(\theta_1, \theta_2) - C(\theta_1, \theta_2')| \leq ?$ and $|C(\theta_1', \theta_2) + C(\theta_1', \theta_2')| \leq ?$ and then use the fact that $\int g(\lambda) d\lambda = 1$ with a positive definite $g(\lambda)$.

Exercise 2: Two-photon interference with entangled photons

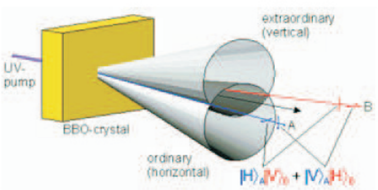
When illuminating a non-linear crystal with a strong pump light field, down-converted photon pairs are created. With type-II phase matching, the photons have orthogonal polarization and are emitted into two cones which are tilted with respect to the pump beam. At the intersection points, the two photons cannot be further distinguished, and the light can essentially be described by an entangled state:

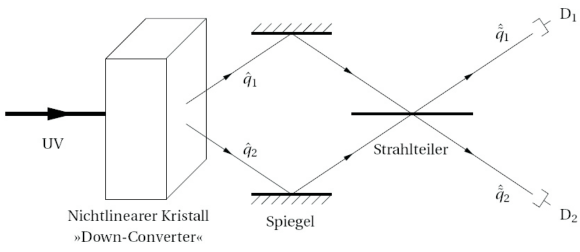
$$|\psi\rangle = \frac{1}{\sqrt{2}} \Big(|H_1, V_2\rangle + e^{i\varphi} |V_1, H_2\rangle \Big), \tag{1}$$

where H and V indicate horizontal and vertical polarization, respectively. The indices $1, 2$ label the two directions. The relative phase φ arises from the crystal birefringence. Using an additional birefringent phase shifter, the value of φ can be set as desired. Similarly, a half wave plate in one path can be used to change horizontal and vertical polarization. On can thus easily produce any of the four Bell states

$$|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}} \Big(|H_1, V_2\rangle \pm |V_1, H_2\rangle \Big), \tag{2}$$

$$|\phi^{\pm}\rangle = \frac{1}{\sqrt{2}} \Big(|H_1, H_2\rangle \pm |V_1, V_2\rangle \Big). \tag{3}$$





In the following the two photon beams are combined at a 50:50 beamsplitter, before they are finally detected by detectors D_1 and D_2.
For each of the four incoming Bell states, calculate the quantum state behind the beamsplitter. What are the coincidences $\langle \hat{n}_1 \hat{n}_2 \rangle$ between detector 1 and 2?