2 Quantization of the Electromagnetic Field

2.1 Basics

Starting point of the quantization of the electromagnetic field are Maxwell's equations in the vacuum (source free):

$$\nabla \cdot B = 0 \tag{1}$$

$$\nabla \cdot D = 0 \tag{2}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{3}$$

$$\nabla \times H = \frac{\partial D}{\partial t} \tag{4}$$

where $B = \mu_0 H$, $D = \varepsilon_0 E$, $\mu_0 \varepsilon_0 = c^{-2}$

In the Coulomb gauge E and B are determined by the vector potential A:

$$B = \nabla \times A \tag{5}$$

$$E = -\frac{\partial A}{\partial t} \tag{6}$$

with the Coulomb gauge condition

$$\nabla \cdot A = 0 \tag{7}$$

one finds

$$\nabla^2 A(r,t) = \frac{1}{c^2}\frac{\partial^2 A(r,t)}{\partial^2 t} \tag{8}$$

and

$$\nabla^2 E(r,t) = \frac{1}{c^2}\frac{\partial^2 E(r,t)}{\partial^2 t} \tag{9}$$

$(\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E\ !)$

The function $A(r,t)$ can be decomposed as

$$A(r,t) = \sum_k c_k u_k(r)\widetilde{a}_k(t) + c_k^* u_k^*(r)\widetilde{a}_k^*(t) \tag{10}$$

or with some convenient normalization (such that the $a_k(t)$ become dimensionless):

$$A(r,t) = -i\sum_k \sqrt{\frac{\hbar}{2\omega_k \varepsilon_0}}\left[u_k(r)a_k(t) + u_k^*(r)a_k^*(t)\right] \tag{11}$$

Plugging this into the wave equation for $A(r,t)$ gives:

$$\left[\nabla^2 + \omega_k^2/c^2\right] u_k(r) = 0 \tag{12}$$

$$\left[\frac{\partial^2}{\partial^2 t} + \omega_k^2\right] a_k(t) = 0 \tag{13}$$

with

$$a_k(t) = a_k e^{-i\omega_k t} \tag{14}$$

$$a_k^*(t) = a_k^* e^{i\omega_k t} \tag{15}$$

one has to find solutions for $u_k(r)$ which can be sinusoidal (e.g. in an optical cavity) or exponential (free runnig waves).

With periodic boundary conditons

$$u_k(r) = u_k(r + L\widehat{x}) = u_k(r + L\widehat{y}) = u_k(r + L\widehat{z}) \tag{16}$$

i.e.

$$u_k(r) = \widehat{\epsilon}_k \frac{1}{\sqrt{V}} e^{ik_n r} \tag{17}$$

or

$$u_k(r) = \widehat{\epsilon}_k \frac{1}{\sqrt{V/2}} \sin(k_n r) \tag{18}$$

where $V = L^3$ and $k_n = 2\pi/L \ (n_x\widehat{x} + n_y\widehat{y} + n_z\widehat{z})$ and the polarization vector $\widehat{\epsilon}_k$ $(\widehat{\epsilon}_k \cdot k_n = 0)$.

Therefore:

$$A(r,t) = -i \sum_k \sqrt{\frac{\hbar}{2\omega_k \varepsilon_0 V}} \widehat{\epsilon}_k \left[a_k e^{-i\omega_k t + ik_n r} + c.c.\right] \tag{19}$$

$$E(r,t) = \sum_k \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} \widehat{\epsilon}_k \left[a_k e^{-i\omega_k t + ik_n r} + c.c.\right] \tag{20}$$

$$H(r,t) = \frac{1}{\mu_0} \sum_k \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} (k_n \times \widehat{\epsilon}_k) \left[a_k e^{-i\omega_k t + ik_n r} + c.c.\right] \tag{21}$$

The normalization constant

$$E_0 = \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} \tag{22}$$

is the *electric field per photon.*

Since the a_k, a_k^* follow the equations of motion of an harmonic oscillator with coordinates:

$$q = \sqrt{\frac{\hbar}{2m\omega}}(a + a^*) \tag{23}$$
$$p = -i\sqrt{\frac{m\omega\hbar}{2}}(a - a^*) \tag{24}$$

the quantization is easily obtained by replacing the c-numbers with operators:

$$a \to \widehat{a} \tag{25}$$
$$a^* \to \widehat{a}^+ \tag{26}$$

which obey the commutation relations

$$\left[\widehat{a}_m, \widehat{a}_n^+\right] = \delta_{mn} \tag{27}$$
$$\left[\widehat{a}_m, \widehat{a}_n\right] = 0 \tag{28}$$
$$\left[\widehat{a}_m^+, \widehat{a}_n^+\right] = 0 \tag{29}$$

In the following the $\hat{}$ above the operators is omitted for clarity.

The Hamiltonian of the quantized free electromagnetic field is thus:

$$H = \frac{1}{2}\int \left(\varepsilon_0 E^2 + \mu_0 H^2\right) \tag{30}$$
$$= \sum_k \hbar\omega_k \left(a_k^+ a_k + 1/2\right) \tag{31}$$

or with the number operator

$$n_k = a_k^+ a_k \tag{32}$$
$$H = \sum_k \hbar\omega_k \left(n_k + 1/2\right) \tag{33}$$

2.2 Number States or Fock States

Number states or Fock states are eigenstates of the number operator $\widehat{n}_k$.:

$$\widehat{n}_k \left|n_k\right\rangle = n_k \left|n_k\right\rangle \tag{34}$$

The operators $\widehat{a}_k$ and $\widehat{a}_k^+$ are called *annihilation and creation operators* and have the following properties:

$$\widehat{a}_k |n_k\rangle = \sqrt{n_k}\, |n_k - 1\rangle \tag{35}$$

$$\widehat{a}_k^+ |n_k\rangle = \sqrt{n_k + 1}\, |n_k + 1\rangle \tag{36}$$

Thus

$$|n_k\rangle = \frac{\left(\widehat{a}_k^+\right)^{n_k}}{(n_k!)^{1/2}} |0\rangle \tag{37}$$

with the *vacuum state* $|0\rangle$.

The energy of a field in a Fock state $|n_k\rangle$ is:

$$\langle n_k| H |n_k\rangle = \sum_{k'} \hbar\omega_{k'} \left(\langle n_k| \widehat{a}_{k'}^+ \widehat{a}_{k'} |n_k\rangle + 1/2\right) \tag{38}$$

$$= \hbar\omega_k n_k + H_0 \tag{39}$$

The expectation value of the electric field of a Fock state vanishes:

$$\langle n_k| E |n_k\rangle = 0 \tag{40}$$

however

$$\langle n_k| E^2 |n_k\rangle = \frac{\hbar\omega_k}{\varepsilon_0 V}(n_k + 1/2) \tag{41}$$

There are non-zero fluctuations even for a vacuum field (*vacuum fluctuations!*)

A problem is the divergence of the energy for the vacuum state:

$$\langle 0| H |0\rangle = \sum_{k'} \frac{1}{2}\hbar\omega_{k'} \to \infty \tag{42}$$

This is not a problem in practise, since experimentally only differences of energies are measured.

Some more properties of Fock states:

- Orthonormality

$$\langle n|m\rangle = \delta_{nm} \tag{43}$$

- Completeness

$$\sum_{n_k=0}^{\infty} |n_k\rangle \langle n_k| = 1 \tag{44}$$

A generalization are multi-mode Fock states:

$$|n_1\rangle\,|n_2\rangle\,...\,|n_l\rangle = |n_1, n_2, ..., n_l\rangle \tag{45}$$

$$a_l\,|n_1, n_2, ..., n_l, ...\rangle = \sqrt{n_l}\,|n_1, n_2, ..., n_l - 1, ...\rangle \tag{46}$$

$$a_l^+\,|n_1, n_2, ..., n_l, ...\rangle = \sqrt{n_l + 1}\,|n_1, n_2, ..., n_l + 1, ...\rangle \tag{47}$$

Any multi-mode state $|\Psi\rangle$ can be written in the Fock representation (i.e. it can be expanded in a Fock state basis):

$$|\Psi\rangle = \sum_{n_1}\sum_{n_2}...\sum_{n_l}...c_{n_1 n_2 ... n_l ...}\,|n_1, n_2, ..., n_l, ...\rangle \tag{48}$$

$$= \sum_{\{n_k\}} c_{\{n_k\}}\,|\{n_k\}\rangle \tag{49}$$

2.3 Coherent States

A special class of states are the so-called *coherent states.*

Definition A: Coherent states are produced by classical light sources:
The classical Interaction Hamiltonian for the interaction of a classical field (described by a classical vector poential) with a current (described by a classical current density $J(r,t)$) is:

$$V_{int} = \int J(r,t)\; A(r,t)\; dr \tag{50}$$

The expression also holds if the classical field is replaced by a quantum field, i.e., a field with the vector potential

$$A(r,t) = -i\sum_k \sqrt{\frac{\hbar}{2\omega_k \varepsilon_0 V}}\,\widehat{\epsilon}_k\left[a_k(t)e^{-i\omega_k t + ik_n r} + c.c.\right] \tag{51}$$

$$= -i\sum_k \frac{1}{\omega_k} E_k\;\widehat{\epsilon}_k\left[a_k(t)e^{-i\omega_k t + ik_n r} + c.c.\right] \tag{52}$$

Plugging this into the Schrödinger equation:

$$\frac{d}{dt}\,|\psi(t)\rangle = -\frac{i}{\hbar}V_{int}\;\;|\psi(t)\rangle \tag{53}$$

and formally integrating yields:

$$|\psi(t)\rangle = \exp\left[-\frac{i}{\hbar}\int_0^t dt' V_{int}(t')\right]|\psi(0)\rangle \ \ e^{i\varphi} \tag{54}$$

The integration is not obvious since $A(r,t)$ and $A(r,t')$ do not commute. However, a correct calculation gives only an additional phase factor!

Therefore:

$$|\psi(t)\rangle = \prod_k \exp\left(\alpha_k a_k^+ - \alpha_k^* a_k\right)|\psi(0)\rangle \tag{55}$$

where

$$\alpha_k = \frac{1}{\hbar\omega_k} E_k \int_0^t dt' \int dr \ \hat{\epsilon}_k \ J(r,t) e^{i\omega_k t' - ikr} \tag{56}$$

If the initial state is the vacuum state ($|\psi(0)\rangle = |0\rangle$) then $|\psi(t)\rangle$ is called a coherent state $|\{\alpha_k\}\rangle$.

$$|\{\alpha_k\}\rangle = \prod_k |\alpha_k\rangle \tag{57}$$

$$|\alpha_k\rangle = \exp\left(\alpha_k a_k^+ - \alpha_k^* a_k\right)|0\rangle_k \tag{58}$$

In the single mode case the operator $D(\alpha)$

$$D(\alpha) = \exp\left(\alpha_k a_k^+ - \alpha_k^* a_k\right) \tag{59}$$

is the *displacement operator*.

$$D(\alpha)|0\rangle = |\alpha\rangle \tag{60}$$

Definition B: A coherent state is an eigenstate of the annihilation operator:

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{61}$$

It is easy to show that $|\alpha\rangle$ can be written in a Fock basis as:

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{62}$$

Since

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle \tag{63}$$

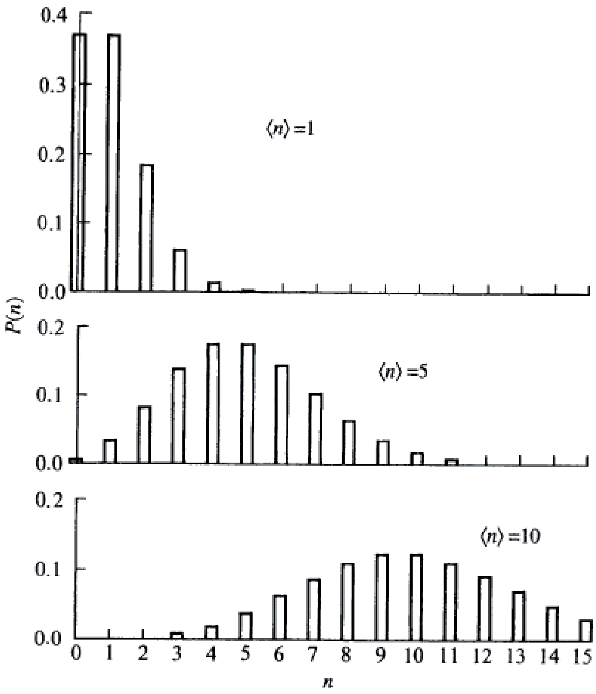

Figure 9: Photon number representation of coherent states for n=1 (a), n=5 (b), and n=10 (c)

it follows

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha a^+} |0\rangle \tag{64}$$

With

$$e^{-\alpha^* a} |0\rangle = |0\rangle \tag{65}$$

the above equation can be written as

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha a^+} e^{-\alpha^* a} |0\rangle \tag{66}$$

Together with the Baker-Hausdorff formula one finally can write:

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha a^+} e^{-\alpha^* a} |0\rangle = |\alpha\rangle = e^{\alpha a^+ - \alpha^* a} |0\rangle = D(\alpha) |0\rangle \tag{67}$$

Definition C: Another way to define the coherent state is to assume that a coherent state should "reproduce a classical state in the best possible fashion", i.e. the mean value of important observables, such as H, E, P, ... should equal the corresponding classical values:

$$\langle\{\alpha\}| H |\{\alpha\}\rangle - H_{vac} = H_{classical}(\{\alpha\}) \tag{68}$$

$$\langle\{\alpha\}| E |\{\alpha\}\rangle = E_{classical}(\{\alpha\}) \tag{69}$$

$$\langle\{\alpha\}| P |\{\alpha\}\rangle = P_{classical}(\{\alpha\}) \tag{70}$$

The equation for the electric field is for example:

$$\langle\{\alpha\}| E |\{\alpha\}\rangle = \langle\{\alpha\}| \sum_k E_k \widehat{\epsilon}_k \left(a_k e^{-i\omega_k t + ikr} + c.c\right) |\{\alpha\}\rangle \tag{71}$$

$$= \sum_k E_k \widehat{\epsilon}_k \left(\alpha_k e^{-i\omega_k t + ikr} + c.c\right) = E_{classical}(\{\alpha\}) \tag{72}$$

Similar equations follow for the other observables.

It is obvious that the equation above holds if $|\alpha\rangle_k$ is an eigenstate of a_k!

To summarize, all definitions of the coherent state are equivalent.
In the next chapter we'll see why the coherent state is called coherent state.

2.4 Properties of Coherent States

- The mean number of photons in a coherent state is

$$\langle\alpha| a^+ a |\alpha\rangle = |\alpha|^2 = \langle n\rangle = \overline{n} \tag{73}$$

The photon number distribution is a Poisson distribution:

$$p(n) = \langle n|\alpha\rangle \langle\alpha|n\rangle = \frac{|\alpha|^{2n} e^{-\overline{n}}}{n!} = \frac{\overline{n}^n e^{-\overline{n}}}{n!} \tag{74}$$

- A coherent state is a minimum uncertainty state.

$$\Delta p \Delta q = \frac{\hbar}{2} \tag{75}$$

This follows from

$$a = \frac{1}{\sqrt{2\hbar\omega}}(\omega q + ip) \tag{76}$$

$$a^+ = \frac{1}{\sqrt{2\hbar\omega}}(\omega q - ip) \tag{77}$$

and

$$
\begin{aligned}
\langle q \rangle &= \sqrt{\frac{\hbar}{2\omega}}\,(\alpha + \alpha^*) && (78)\\
\langle p \rangle &= \sqrt{\frac{\hbar\omega}{2}}\,(\alpha - \alpha^*) && (79)\\
\left\langle p^2 \right\rangle &= \frac{\hbar\omega}{2}(\alpha^2 + \alpha^{*2} + 2n + 1) && (80)\\
\left\langle q^2 \right\rangle &= \frac{\hbar}{2\omega}(\alpha^2 + \alpha^{*2} + 2n + 1) && (81)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(\Delta p)^2 &= \left\langle p^2 \right\rangle - \langle p \rangle^2 && (82)\\
(\Delta q)^2 &= \left\langle q^2 \right\rangle - \langle q \rangle^2 && (83)\\
\Delta p \Delta q &= \hbar/2 && (84)
\end{aligned}
$$

- The set of coherent states is a complete set:

$$
\frac{1}{\pi}\int |\alpha\rangle\langle\alpha| \; d^2\alpha = 1 \qquad (85)
$$

- Two coherent states are not orthogonal:

$$
\begin{aligned}
\langle \alpha | \alpha' \rangle &= \exp\left(-\frac{1}{2}|\alpha|^2 + \alpha'\alpha^* - \frac{1}{2}|\alpha'|^2\right) && (86)\\
|\langle \alpha | \alpha' \rangle| &= \exp\left(-|\alpha - \alpha'|^2\right) && (87)
\end{aligned}
$$

If $|\alpha - \alpha'|$ is very large then the two states are "nearly" orthogonal.

Coherent states are overcomplete (every state can be expanded in the $\{|\alpha\rangle\}$-basis, but not in a unique way.

2.5 Thermal State

A third class of quantum states of light are thermal states. These states are produced by thermal sources, e.g. a light bulb or a discharge lamp.

The photon number distribution of a thermal state is:

$$p(n) = \frac{1}{1+\overline{n}} \left(\frac{\overline{n}}{1+\overline{n}} \right)^n \tag{88}$$

This follows directly from the Boltzmann distribution:

$$p(n) = \frac{\exp\left(-E_n/k_B T\right)}{\sum_n \left(-E_n/k_B T\right)} \tag{89}$$

The mean number of photons in a thermal state obeys a Planck-distribution:

$$p(n) = \frac{1}{e^{\hbar\omega/k_B T} - 1} \tag{90}$$

The density operator of a thermal state is:

$$\rho_{thermal}(n) = \sum_n \frac{\overline{n}^n}{\overline{n}^{n+1}} |n\rangle\langle n| \tag{91}$$

Figure 10 shows the photon number distribution for a thermal and a coherent state. Obviously, the thermal state has a wider photon number distribution.

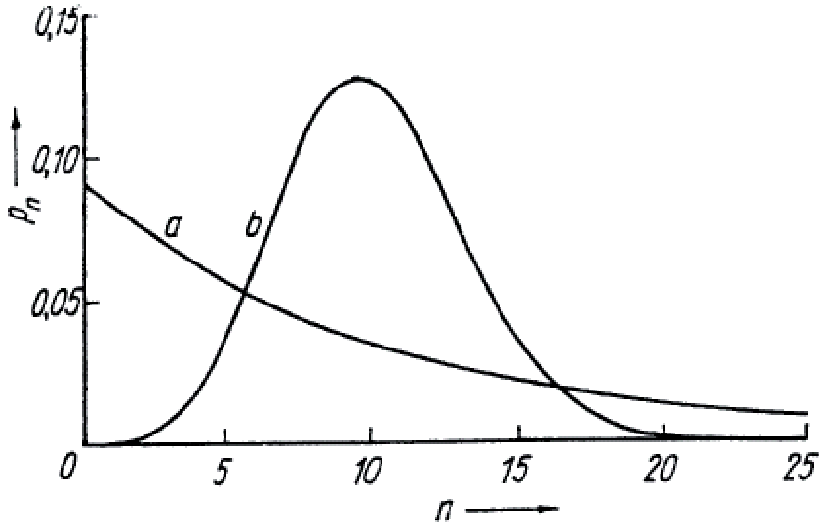

Figure 10: Photon number representation of a thermal state (a) and a coherent state (b) for $\overline{n} = 10$