

6 Entanglement

6.1 Basic Principles

Entangled states are very interesting states because they exhibit correlations that have no classical analog. They are of particular importance in quantum computation and quantum information. As an example let us take the entangled bi-partite pure state:

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle_{AB} + |10\rangle_{AB}) \in H_A \otimes H_B$$

This state can not be decomposed as a simple product state $|q\rangle_A |p\rangle_B$ for $q, p \in \{0, 1\}$. The state is maximally entangled, i.e. when we trace over the state B then the reduced density operator ρ_A of the system will be a multiple of the identity operator. This means that if we measure in system A in **any basis** the result will be completely random (0 or 1 with equal probability 1/2). However, there is a perfect correlation: Whenever we measure 1 in system A then we will measure 0 in system B with certainty and vice versa. $|\psi^+\rangle$ is one of the so-called four Bell states:

$$\begin{aligned} |\psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle_{AB} + |10\rangle_{AB}) \\ |\psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle_{AB} - |10\rangle_{AB}) \\ |\phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB}) \\ |\phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle_{AB} - |11\rangle_{AB}) \end{aligned}$$

They form a convenient basis of bi-partite quantum states of two-dimensional Hilbert spaces.

6.2 EPR Paradoxon

In the 20s and 30s it became evident that some properties in quantum mechanics can be assigned only to the quantum mechanical system, but not necessarily to its

constituents. This led Einstein, Podolsky and Rosen (EPR) to their remarkable 1935 paper where they concluded that quantum mechanics is not a complete theory of nature (EPR paradox). The conclusion was derived from some common sense requirements that EPR postulated:

1. Completeness: Each element of realism should have its correspondence in a theory.
2. Realism: If a property can be assigned to a physical system with certainty then there exists an element of realism that corresponds to this property.
3. Locality: Measurements of different elements of realism in spatially separated systems can not influence each other.

The discussion between EPR and e.g. Niels Bohr was considered a merely philosophical one until in 1964 John Bell suggested to measure certain correlations between entangled entities. These correlations would violate an inequality (Bell's inequality) if quantum mechanics was correct.

To derive Bell's inequalities let us consider a spin 1 particle that decays into two spin 1/2 particles. Spin is an example for a dichotomic variable, i.e. a variable that can take only two values (*up* or *down* or +1 or -1). Let us now consider correlations between measurements of the spin along a particular axis in system A (variable $a(\alpha)$) and system B (variable $b(\beta)$), respectively. It is easy to show that:

$$\begin{aligned} a(\alpha_1)b(\beta_1) + a(\alpha_2)b(\beta_1) + a(\alpha_1)b(\beta_2) - a(\alpha_2)b(\beta_2) = \\ (a(\alpha_1) + a(\alpha_2))b(\beta_1) + (a(\alpha_1) - a(\alpha_2))b(\beta_2) = \pm 2. \end{aligned}$$

Now if the theory is not complete there are some hidden parameters λ that describe the outcome of each measurement. If we do not know λ then we have to average over all possible λ which follow the probability distribution $p(\lambda)$ and find:

$$\left| \int P(\lambda) \{ (a_1(\lambda) + a_2(\lambda))b_1(\lambda) + (a_1(\lambda) - a_2(\lambda))b_2(\lambda) \} \right| \leq 2$$

The quantum mechanical expression is:

$$\langle a_1 b_1 \rangle + \langle a_2 b_1 \rangle + \langle a_1 b_2 \rangle - \langle a_2 b_2 \rangle \stackrel{\text{for a special case}}{=} 2\sqrt{2}$$

where e.g. $a_1 = a(\alpha_1) = \alpha_1 \cdot \sigma^A$ is the spin projection of the particle A on the axis α_1 and $\langle \rangle$ denotes the expectation value. It can be shown easily that this expression gives the value $2\sqrt{2}$ for an angle of $\frac{\pi}{4}$ between the axes $(\alpha_i \beta_j)$ and the entangled state $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$. Thus, the Bell's inequality is violated by entangled states. There are various forms of Bell's inequalities. However, it can be shown that any pure entangled state violates Bell's inequalities for certain parameter settings.

Meanwhile a lot of experiments have been performed (most of them with polarization entangled photon pairs). Today experiments can be performed in minutes which violate Bell's inequalities by many standard deviations.

The deviations between classical and quantum correlations become even more pronounced if more than two particles are involved. As an example it has been proposed and demonstrated by (Greenberger, Horne and Zeilinger) that the three-particle entangled state:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|111\rangle + |000\rangle)$$

provides a correlation event between detector clicks (for the three particles) that *never* occur in a classical experiment. Thus, the deviation between the quantum and the classical world could in principle be measured in a single event and not in a statistical measurement outcome as in the case of Bell's inequality.

For mixed states the entanglement is much more subtle. A lot of theoretical work is still in progress in order to understand and define the concept of entanglement of arbitrary many-particle state.

6.3 Experimental Tests of Bell's inequality

6.3.1 Requirements

An experimental test of Bell's inequality already requires some basic tools of quantum state manipulation that are also necessary ingredients for quantum computing:

1. Production of an entangled state of the form:

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle_{AB} + |10\rangle_{AB}) \in H_A \otimes H_B$$

2. Manipulation of the state (e.g. by optical elements like polarizers or beam splitters for photon states)
3. Detection with very high efficiency.

6.3.2 Entangled State Production

First Gedankenexperiments considered the decay of elementary particles into sub-constituents as a possible source for entangled states. An example is the positronium decay, i.e. a decay of a spin 0 particle into two spin 1/2 particles with opposite momentum. However, the main system for entangled states today are entangled photons. We consider two ways to produce entangled photon states:

1. Spontaneous cascade decay in a single atom.

In an atom an excited electron may decay in a cascade via an intermediate state. If the cascade is from a state with total angular momentum $J_1 = 0$ via a state with $J_2 = 1$ to a final state $J_1 = 0$, then the emitted two photons are in a polarization entangled state:

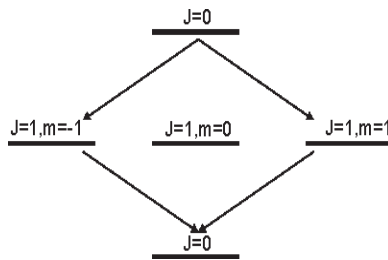


Figure 34: Schematics of Cascade Decay in an Atom

2. Spontaneous parametric down-conversion.

This process has already been discussed in the previous chapter. It is depicted on the right of the following cartoon.



Figure 35: Up- and Down-Conversion [from Fundam. of Photonics, Saleh, Teich]

3. With:

$$\omega_0 = \omega_1 + \omega_2 \quad \text{Frequency-Matching Condition}$$

$$k_0 = k_1 + k_2 \quad \text{Phase-Matching Condition}$$

There are two different types of down-conversion. In **type-I down-conversion** both photons ω_1 and ω_2 have the same polarization. The two photons emerge on two concentric cones symmetric with respect two the axis of the so-called pump beam of frequency ω_0 . In **type-II down-conversion** both photons ω_1 , ω_2 have opposite (horizontal and vertical) polarization.

6.4 Three experiments

6.4.1 Aspect's Experiment:

First experimental test by Freedman and Clauser in 1972. The experiment from Alain Aspect in 1982 attacked one *loophole*.

Loophole = a way to "save" hidden-variable theory.

Aspect attacked the *communication-loophole*, i.e., the position of polarizers was fixed in previous experiments. In principle there could have been a mechanism that *informs* the photons from the source about the position and thus marks them with a certain variable.

Detection-loophole: not all photons are detected, the undetected ones could have behaved in a different way!

The following picture gives the experimental setup of Aspect's experiment.

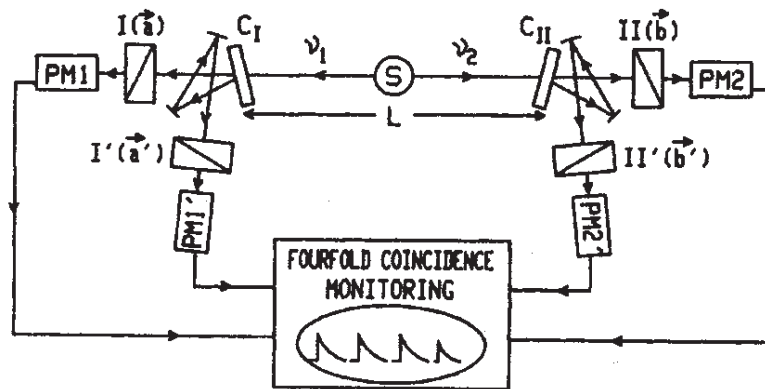


Figure 36: Setup of Aspect's Experiment [from PRL 49, 1804 (1982)]

Aspect used a slightly different Bell's inequality. The outcome of the measurement after about 12000 seconds was:

$$\begin{aligned}
 S_{LHVT} &\leq 0 \\
 S_{QM} &= 0.112 \\
 S_{EXP} &= 0.102 \pm 0.020
 \end{aligned}$$

where the first row gives a boundary for a local hidden-variable theory, the second row the quantum mechanical prediction for Aspect's experiment and the last row the experimental result.

6.4.2 The Experiment by Ou and Mandel

Ou and Mandel used parametric down-conversion in a non-linear crystal to demonstrate the violation of Bell's inequalities (Ou and Mandel, Phys. Rev. Lett. 61, 50 (1988)).

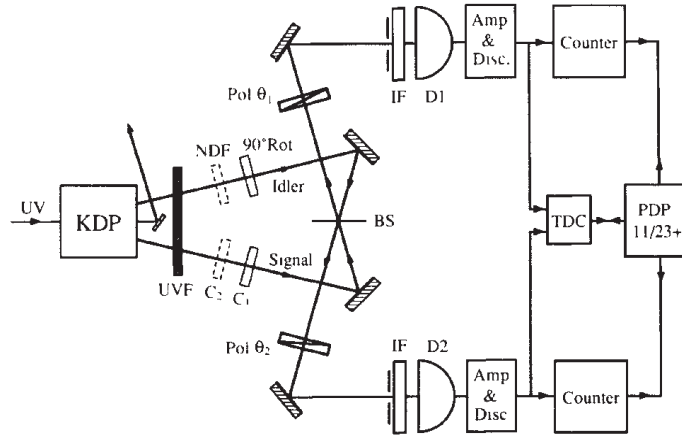


Figure 37: Experimental apparatus in the Ou and Mandel experiment [from Ou and Mandel, Phys. Rev. Lett. 61, 50 (1988)]

It is interesting to note that this experiment did not use the potential of entangled state generation in the down-conversion process yet. It rather used the fact that a product state of the form

$$|\psi_0\rangle = |1\rangle_x |1\rangle_y \quad (247)$$

can be created at the output of the crystal (with the help of the 90° rotation), where x , and y denote two orthogonal linear polarisations.

Then, this state falls on a beam splitter. After the beam splitter the state is:

$$|\psi_0\rangle = \frac{1}{2} [|X\rangle_1 |Y\rangle_2 + |Y\rangle_1 |X\rangle_2 - i |X\rangle_1 |Y\rangle_1 + i |X\rangle_2 |Y\rangle_2] \quad (248)$$

If only coincidences are measured afterwards, then the last two terms do not contribute (both photons in the same arm) and can be discarded.

Thus the *post-selected* state is

$$|\psi_0\rangle \stackrel{ps}{=} \frac{1}{2} |X\rangle_1 |Y\rangle_2 + |Y\rangle_1 |X\rangle_2 \quad (249)$$

an entangled state.

This state was used to measure Bell's inequalities. A deviation of 6 standard deviations was measured in a few minutes.

6.4.3 Zeilinger's Experiment:

Kwiat et al. introduced a clever method to produce entangled photons from a down-conversion crystal right away without postselection and without loss of 50% of the wavefunction:

They used off-axis type-II down-conversion: The ordinary and extraordinary beams are emitted on cones, which are slightly tilted with respect to the pump beam as depicted in the following cartoon:

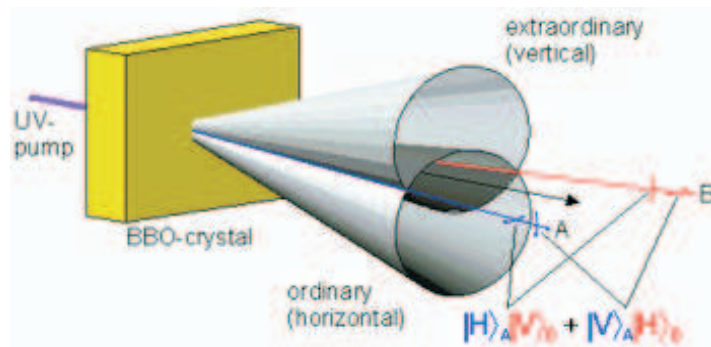


Figure 38: Type-II Down Conversion [from <http://www.quantum.univie.ac.at/>]

At the intersection points the light is unpolarized. At these points, there is no "labeling" of the polarization due to the beam position. The photon pairs are in an entangled state! Only photons from the intersection of the two cones are collected in an experiment and provide the polarization entangled state:

$$|\psi^+\rangle = \frac{1}{\sqrt{2}} (|HV\rangle_{AB} + |VH\rangle_{AB}) \in H_A \otimes H_B$$



Figure 39: Photo of Photons from type-II down conversion in a BBO crystal [from <http://www.quantum.univie.ac.at/>]

The brightness of such a source could be as high as $5 \cdot 10^5$ pairs per second.

Problem: Due to the birefringence the photons have slightly different velocities. The time delay between two photons created at the two faces of the crystal is

$$\delta T = L(1/v_o - 1/v_e) \quad (250)$$

where L is the crystal length and v_o , and v_e are the group velocities of the ordinary and extraordinary beams, respectively.

Accordingly, a lateral displacement results by the tilting of the beam with respect to each other:

$$d = L \tan \vartheta \quad (251)$$

If the coherence time of the photons is shorter than δT or if the coherence width is smaller than d , then the polarization of the photons is "labelled" in time and position, respectively.

Usually, d is much smaller than the pump beam diameter, but δT has to be compensated by an additional birefringent crystal.

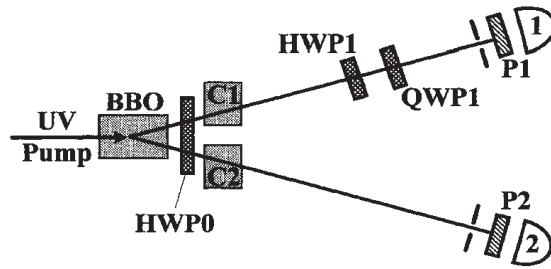


Figure 40: Principle of walk-off compensation [from Kwiat et al., Phys. Rev. Lett. 75, 4337 (1995)]

With this source of entangled photons coincidence count rates of the order **1500 ct/sec** were obtained, whereas modern sources today achieve **>10.000 ct/sec**.

With this source another loophole was closed in Zeilinger's experiment: One remaining problem in Aspect's experiment was that the position of the polarizers was changed periodically and thus in a predictable way. The experiment performed by the group of Zeilinger used two detectors for analysis of the entangled photons that were separated spacelike. That means if Einstein-causality holds and if the polarizer setting is changed fast enough then there is no way whatsoever for the two photons to exchange information about the polarizers.

In Zeilinger's experiment the source of entangled photons was the process of type-II downconversion in a non-linear crystal. The photons were transmitted through optical fibers to two buildings on the campus of Innsbruck University 400 meters apart and were detected by APDs. An electro-optic modulator tilted the polarization angle in each arm between two different position (similar as in Aspect's experiment). A physical random-generator was used to tell the modulator when to switch. The detectors were synchronized with the help of an atomic clock in order to determine the correlations. The outcome of the experiment after only 10 seconds was (where the same form of Bell's inequality was used as in this script):

$$\begin{aligned}
 S_{LHVT} &\leq 2 \\
 S_{QM} &= 2.74 \\
 S_{EXP} &= 2.73 \pm 0.02
 \end{aligned}$$

Thus a deviation of more than 30 standard deviations was detected within seconds!

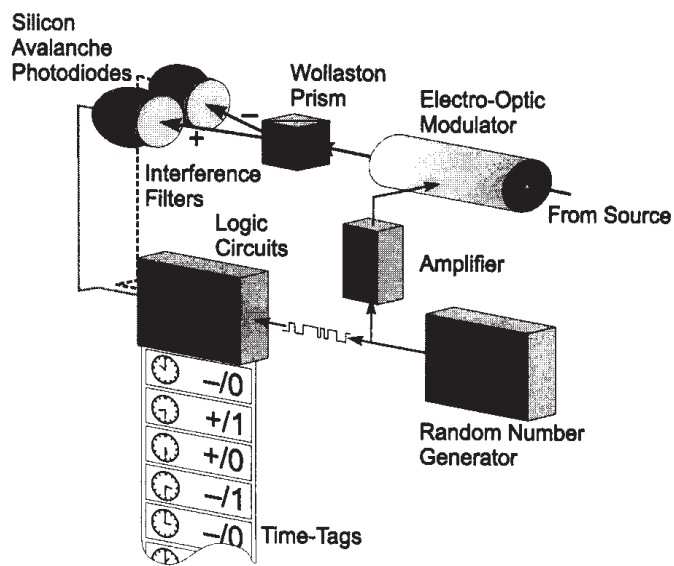


Figure 41: Experimental Setup of Zeilinger's experiment (One Detector Arm). [from <http://www.quantum.univie.ac.at/>]

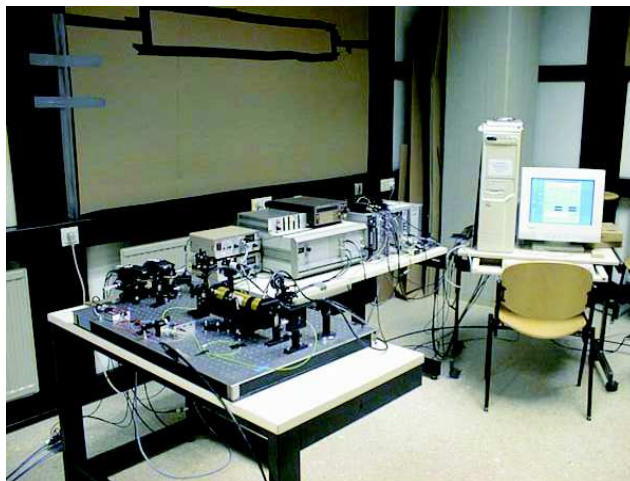


Figure 42: One detector station in Zeilinger's experiment. [from <http://www.quantum.univie.ac.at/>]

6.5 Single Photon Detectors

Detectors for single photons require a very high efficiency and low dark counts to establish a large signal-to-noise ratio. In present day experiments **avalanche photodiodes** (APDs) are used as single photon detectors. An APD utilizes the photo effect in a semiconductor (e.g. silicon).

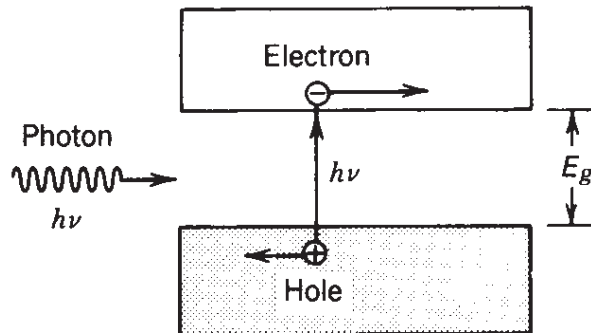


Figure 43: Absorption of a Photon in a Semiconductor [from Fundamentals of Photonics, Saleh, Teich]

A single photon is absorbed and creates two photoexcited carriers (an electron and a hole). An applied electric field induces these carriers to move, which results in a circuit current. In an APD the electric field is large enough that the carriers gain enough energy to ionize additional carriers via impact ionization. The so-created avalanche of carriers is detected as a current pulse. An APD is realized as a special $p-i-n$ silicon reverse-biased photodiode. The intrinsic (undoped) layer in-between the $n-$ and $p-$ doped region has the following advantages:

- The increased depletion layer of the device provides an increased area where photons can be captured.
- The increased width of the intrinsic layer reduces the capacity and therefore the RC time constant.

- The reduced ratio between diffusion length and drift length results in a greater proportion of the generated current being carried by the faster drift process. This reduces the noise-factor and leads to improved stability. (Avalanches tend to be unstable if both carriers substantially contribute!)

The following figures show the responsivity of a commercially available $p-i-n$ photodiode.

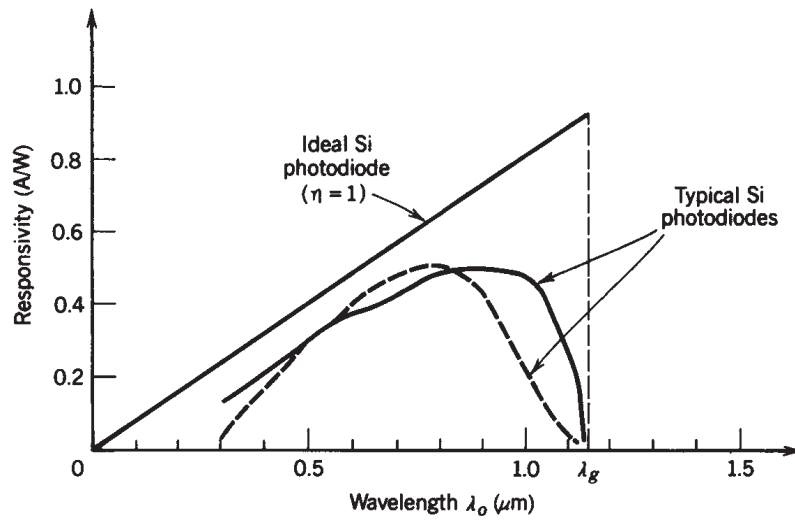


Figure 44: Responsivity of a commercially available p-i-n photodiode [from Fundamentals of Photonics, Saleh, Teich]

Schematic of the multiplication process in an APD and typical APD structure:

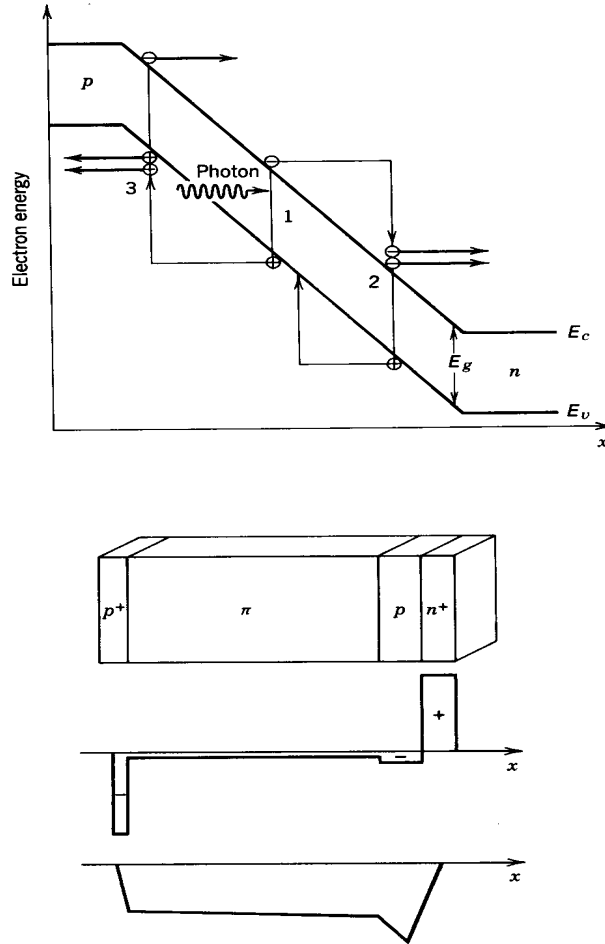


Figure 45: Top: Schematics of the Avalanche Process in APD; Bottom: Structure (top), Charge Density (middle), and Electric Field (bottom) of an APD [from Fundamentals of Photonics, Saleh, Teich]

The **quantum-efficiency** which is the probability that a photon impinging on a photodiode creates an electron-hole pair can exceed 90%. However, the probability to detect a single photon eventually as an electric pulse is something like 60%-70%. An important issue apart from the detection efficiency is the noise property of a photodetector. Noise can have various origins:

- **Photon Noise:**

Photons can be considered as a stream of particles which arrive randomly.

- **Photoelectron Noise:**

There is only a finite probability (quantum-efficiency) that a photon produces an electron-hole pair. This randomness introduces noise.

- **Gain Noise:**

The internal gain in an APD is random, i.e. each electron-hole pair creates a random number of additional random carriers. This additional noise is described in terms of an *Excess Noise Factor* F . This effect is more pronounced if both types of carriers produce an avalanche. In an APD the height distribution of electric pulses at the output has an exponential distribution. It is thus not possible to discriminate real pulses from internal noise (dark counts). Furthermore, it is not possible to discriminate between a single photon and two or more photons at a time. Recently, detectors have been developed which give a noise free amplification ($F = 1$), and which can thus discriminate between a single photon and two or more photons.

The following picture shows the signal to noise-ratio versus the gain for different ratios k of ionized electrons and ionized holes.

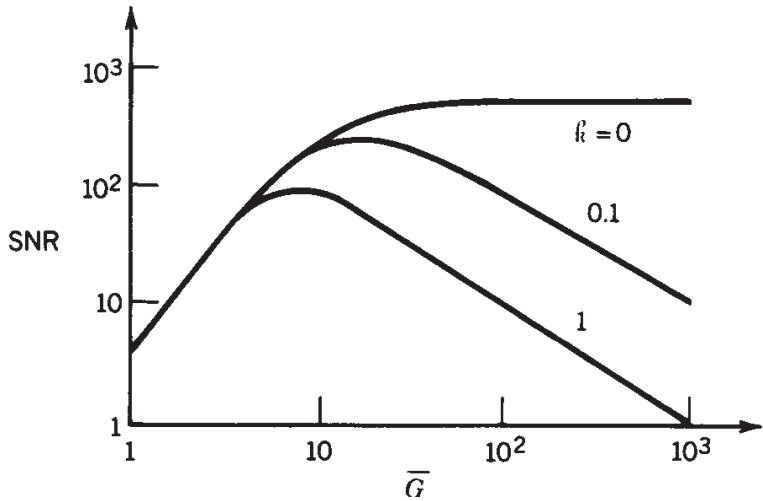


Figure 46: Signal-to-Noise Ratio versus Gain of an APD [from Fundamentals of Photonics, Saleh, Teich]