# 7 Quantized Interaction of Light and Matter

## 7.1 The electron wavefunction

The wavefunction of an electron $\Psi(x)$ can be decomposed with a complete set of eigenfunctions $\psi_j(x)$ which obey the Schroedinger equation:

$$H_0\psi_j(x) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi_j(x) = E_j\psi_j(x) \tag{252}$$

In analogy to the quantization of the light field one can write:

$$\Psi(x) = \sum_j b_j^+\psi_j(x) \tag{253}$$

with the *fermionic* creation operator $b_j^+$.

The anti-commutation relation of the fermionic creation and annihilation operator are:

$$\{b_i, b_j\} = \left\{b_i^+, b_j^+\right\} = 0 \tag{254}$$
$$\left\{b_i, b_j^+\right\} = 1 \tag{255}$$

An arbitrary state can thus be constructed by applying $b_j^+$ operators to the vacuum:

$$|\{j\}\rangle = b_{j1}^+ b_{j2}^+ ... b_{jn}^+ |0\rangle \tag{256}$$

Due to the fermionic nature:

$$(b_j^+)^2 |0\rangle = 0 \qquad \text{or more general} \qquad (b_j^+)^2 |\varphi\rangle = 0 \tag{257}$$

The expectation value for the atomic Hamiltonian $H_0$

$$H_0 = \sum_j b_j^+ b_j E_j \tag{258}$$

is

$$\langle\psi| H_0 |\psi\rangle = \sum_j E_j \tag{259}$$

Since a lot of problems in quantum optics deal with the simplified case of *two-level atoms* it is convenient to limit the atomic Hilbert space to two dimensions and to introduce the Pauli spin operators $\sigma_j \in H^{\oplus 2}$ (similar as in a single spin system):

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{260}$$

Together with the raising and lowering operators

$$\sigma^+ = \frac{1}{2}(\sigma_x + i\sigma_y); \quad \sigma^- = \frac{1}{2}(\sigma_x - i\sigma_y) \tag{261}$$

The latter operators have the following properties:

$$\left[\sigma^+, \sigma^-\right] = 2\sigma_z; \qquad \left[\sigma^\pm, \sigma_z\right] = \mp\sigma^\pm; \qquad \left\{\sigma^+, \sigma^-\right\} = 1 \tag{262}$$

### 7.2 Bloch representation

If we assume a two-level system of two atomic states $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then the following correspondence holds:

| pseudo-spin operators | electron operators | |
|---|---|---|
| $\sigma^+$ | $b_1^+ b_2$ | $\lvert 1\rangle \langle 2\rvert$ |
| $\sigma^-$ | $b_2^+ b_1$ | $\lvert 2\rangle \langle 1\rvert$ |

Any state of the two-level atom can be written as:

$$|\psi\rangle = c_1 |1\rangle + c_2 |2\rangle \qquad \text{with} \qquad |c_1|^2 + |c_2|^2 = 1 \tag{263}$$

More generally (non-pure states) one has to write down the density operator $\rho$:

$$\rho^{(A)} = \rho_{11} |1\rangle \langle 1| + \rho_{22} |2\rangle \langle 2| + \rho_{12} |1\rangle \langle 2| + \rho_{21} |2\rangle \langle 1| \tag{264}$$

$$\text{where} \quad \rho_{ij} = \left\langle c_i c_j^* \right\rangle \quad i, j = 1, 2 \tag{265}$$

$\rho$ has a representation in terms of a two-dimensional Hermitian covariant matrix.

The *Bloch-representation* has a very intuitive geometrical representation of the state.

Definition of the Bloch-vector $\overrightarrow{r}$:

$$r_1 = 2\,\mathrm{Re}(\rho_{12}) \tag{266}$$
$$r_2 = 2\,\mathrm{Im}(\rho_{12}) \tag{267}$$
$$r_3 = \rho_{22} - \rho_{11} \tag{268}$$

Therefore:

$$|1\rangle \triangleq (0, 0, -1) \tag{269}$$
$$|2\rangle \triangleq (0, 0, 1) \tag{270}$$

The Bloch-vector for a pure state lies on a sphere of radius $|r| = 1$.

Generally, it follows:

$$r_1^2 + r_2^2 + r_3^2 = 4\,|\rho_{12}|^2 + |\rho_{22} - \rho_{11}|^2 \tag{271}$$
$$= 1 - 4\left(\rho_{22}\rho_{11} - |\rho_{12}|^2\right) \tag{272}$$

from the Cauchy-Schwartz inequality on finds:

$$\rho_{22}\rho_{11} - |\rho_{12}|^2 = \left\langle |c_2|^2 \right\rangle \left\langle |c_1|^2 \right\rangle - |\langle c_1 c_2^* \rangle|^2 \geq 0 \tag{273}$$

and thus

$$r_1^2 + r_2^2 + r_3^2 \leq 1 \tag{274}$$

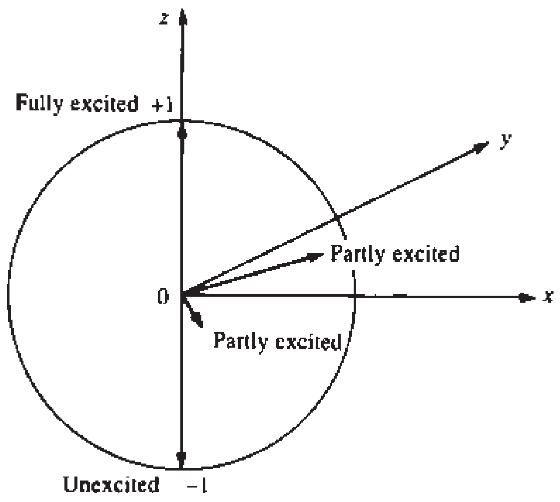


Figure 47: Bloch representation of a state of a two-level atom [from Mandel "Optical Coherence and Quantum Optics"]

### 7.3 Interaction of an atom with a classical field

The interaction of a classical field $E(t)$ with an atom can be described via the dipole interaction:

$$H_I = -\overrightarrow{\mu}(t) \cdot \overrightarrow{E}(t) \tag{275}$$

The time evolution of the density matrix $\rho(t)$ describing the state of the atom (Hamiltonian $H_A = \frac{1}{2}\hbar\omega_0\sigma_z$) follows from the Schroedinger equation with the Hamiltonian $H = H_A + H_I$:

$$\frac{\partial\rho(t)}{\partial t} = \frac{1}{i\hbar}\left[H_A + H_I, \rho(t)\right] \tag{276}$$

The general form of this equation of motion is:

$$\dot{\rho}_{11} = \frac{1}{i\hbar}\left[\langle 1|\, H_I \,|2\rangle\, \rho_{21} - c.c\right] \tag{277}$$

$$\dot{\rho}_{22} = -\frac{1}{i\hbar}\left[\langle 1|\, H_I \,|2\rangle\, \rho_{21} - c.c\right] \tag{278}$$

$$\dot{\rho}_{12} = \frac{1}{i\hbar}\left[-\hbar\omega_0\rho_{12} + \langle 1|\, H_I \,|2\rangle\, (\rho_{22} - \rho_{11})\right] \tag{279}$$

$$\dot{\rho}_{21} = \frac{1}{i\hbar}\left[\hbar\omega_0\rho_{21} + \langle 2|\, H_I \,|1\rangle\, (\rho_{11} - \rho_{22})\right] \tag{280}$$

Obviously $(\dot{\rho}_{11} + \dot{\rho}_{22}) = 0$.

Remark: The link to classical or semiclassical physics is via the polarisation

$$P = \langle 1|\, H_I \,|2\rangle\, \rho_{12} + c.c. \tag{281}$$

These equations of motions can be expressed by the Bloch vector and are called *Bloch equations*:

$$\dot{r}_1 = \frac{1}{\hbar} 2\,\mathrm{Im}\left[\langle 1|\, H_I \,|2\rangle\right] r_3 - \omega_0 r_2 \tag{282}$$

$$\dot{r}_2 = -\frac{1}{\hbar} 2\,\mathrm{Re}\left[\langle 1|\, H_I \,|2\rangle\right] r_3 + \omega_0 r_1 \tag{283}$$

$$\dot{r}_3 = -\frac{2}{\hbar}\,\mathrm{Im}\left[\langle 1|\, H_I \,|2\rangle\right] r_1 + \frac{2}{\hbar}\,\mathrm{Re}\left[\langle 1|\, H_I \,|2\rangle\right] r_2 \tag{284}$$

Obviously $d/dt(r_1^2 + r_2^2 + r_3^2) = 0!$

The motion of the Bloch vector can be described as a (complicated) precession around a vector $Q(t)$:

$$\frac{d}{dt}\overrightarrow{r} = Q \times \overrightarrow{r} \tag{285}$$

with

$$Q = \begin{pmatrix} \frac{2}{\hbar}\,\mathrm{Re}\,\langle 1|\, H_I \,|2\rangle \\ \frac{2}{\hbar}\,\mathrm{Im}\,\langle 1|\, H_I \,|2\rangle \\ \omega_0 \end{pmatrix} \tag{286}$$

If the interaction with a classical single-mode field $E(t) = \hat{\epsilon} E_0(t) \exp(-i\omega_1 t) + c.c.$ is evaluated then the term $\langle 1| H_I |2\rangle$ becomes:

$$\langle 1| H_I |2\rangle = -\overrightarrow{\mu}_{12} \overrightarrow{E}(t) = -\langle 1| \overrightarrow{\mu} |2\rangle \overrightarrow{E}(t) \tag{287}$$

The fast rotation of the Bloch-vector around the z-axis at the optical frequency $\omega_0$ can be eliminated by transforming into a rotating frame:

$$\overrightarrow{r}' = \Theta \cdot \overrightarrow{r} \tag{288}$$

with

$$\Theta = \begin{pmatrix} \cos\omega_1 t & \sin\omega_1 t & 0 \\ -\sin\omega_1 t & \cos\omega_1 t & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{289}$$

This leads to the Bloch equations in the rotating frame:

$$\dot{r}_1' = (\omega_1 - \omega_0) r_2' \tag{290}$$

$$\dot{r}_2' = (\omega_0 - \omega_1) r_1' + \Omega r_3' \tag{291}$$

$$\dot{r}_3' = -\Omega r_2' \tag{292}$$

with the *Rabi frequency* $\Omega$:

$$\Omega = 2\overrightarrow{\mu}_{12} \hat{\epsilon} \left| E_0(t) \right| / \hbar \tag{293}$$

One can also write

$$\dot{\overrightarrow{r'}} = Q' \times \overrightarrow{r'} \quad \text{with} \quad Q' = \begin{pmatrix} -\Omega \\ 0 \\ \omega_0 - \omega_1 \end{pmatrix} \tag{294}$$

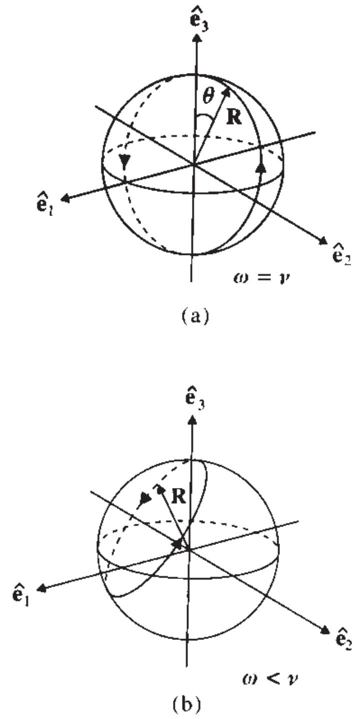


Figure 48: Precession of Bloch vector vor $\delta = 0$ (a) and $\delta \neq 0$ (b) [from Scully "Quantum Optics"]

### 7.4 Ramsey fringes

If the field is in resonance with the atomic transition $(\omega_1 - \omega_0) = 0$ then it is:

$$r_1'(t) = 0 \tag{295}$$
$$r_2'(t) = -\sin \Omega t \tag{296}$$
$$r_3'(t) = \cos \Omega t \tag{297}$$

A pulse which is applied to the atom initially in the ground state ($r = (0, 0, 1)$) which has the pulse area $\Omega t = \pi$, a so-called $\pi$-*pulse*, flips the atomic state to the excited state, whereas a pulse with area $\Omega t = \pi/2$, a $\pi/2$-*pulse*, creates a coherent superposition of upper and lower atomic state of equal weight:

$$
\begin{array}{lll}
\Omega t = \pi & \pi\text{-pulse} & |2\rangle \longrightarrow |1\rangle \\
\Omega t = \pi/2 & \pi/2\text{-pulse} & |2\rangle \longrightarrow (|2\rangle + |1\rangle)/\sqrt{2}
\end{array}
\qquad (298)
$$

A small detuning $\delta = \omega_1 - \omega_0$ leads to a rotation of the Bloch vector in the x-y-plane if there is a non-zero component of $r_1$ or $r_2$.

A method to exploit this effect in order to perform precise measurements of a frequency $\omega$ was proposed by Ramsey, who was awarded the Nobel prize for this idea in 1989:

- First a $\pi/2$-pulse is applied to an atom, which is initially in the ground state. This flips the Bloch vector into the x-y-plane.
- If there is no detuning (e.-mag. field in exact resonance with the atomic transition) then a second $\pi/2$-pulse after some time $T$ flips the Bloch vector exactly to the excited state, which can then be detected.
- If, however, there is some detuning then the Bloch vector rotates in the x-y-plane by an angle $\delta \cdot T$. A second $\pi/2$-pulse would then usually not tilt the Bloch vector exactly to the excited state (in the extreme case the Bloch vector may even be tilted back to the ground state).

This method can be used to compare the frequency of a field to an atomic transition and is called *Ramsey-method*. By changing $T$ or $\delta$ the probability to detect the atom in the excited state oscillates. These oscillations are also called *Ramsey fringes*.

In a Ramsey interferometer the two pulses have to be separated in time as far as possible to obtain highest sensitivity. The sensitivity is not limited by the time-of-flight of the atom through a single interaction zone in the experiment.

Modern atomic clocks (e.g. Cs clocks) use the Ramsey method to stabilze an RF-field to a narrow atomic transition.

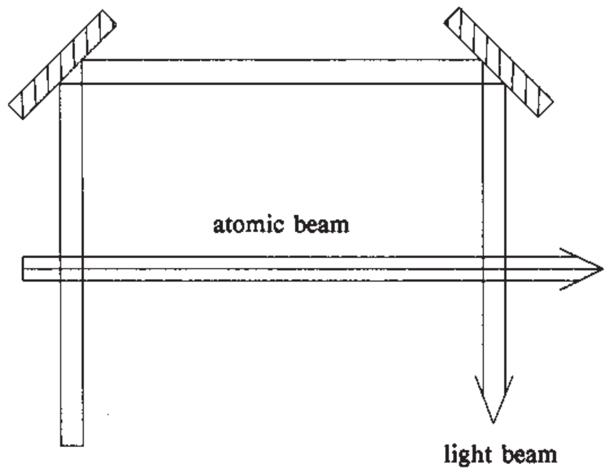


Figure 49: Principle setup for a Ramsey measurement.

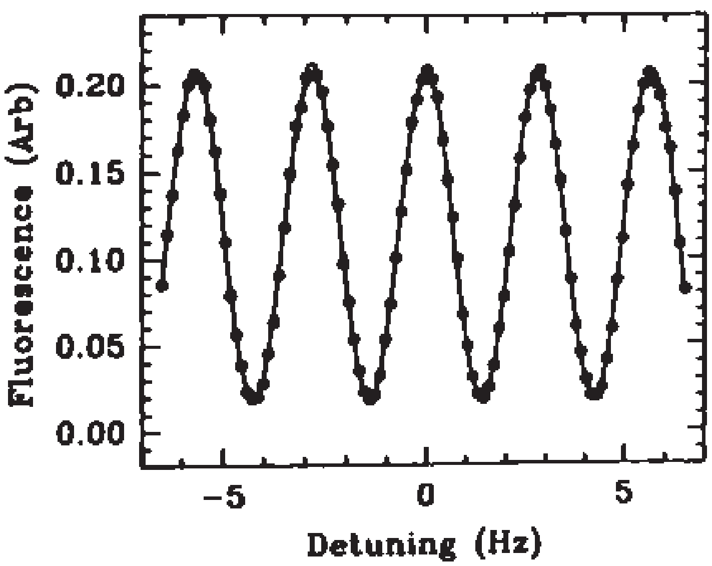


Figure 50: Ramsey fringes

### 7.5 Interaction of an atom with a quantized field

In order to describe the interaction of an atom with a quantum field it is convenient to start from the Hamiltonian:

$$H = \frac{1}{2m}(p - eA)^2 + eV(x) + H_{field} \tag{299}$$

$$= H_A + H_I + H_{field} \tag{300}$$

where

$$H_A = \int \psi^+(x) \left( -\frac{\hbar^2}{2m}\nabla^2 + eV(x) \right) \psi(x)\, dx \tag{301}$$

$$H_I = \int \psi^+(x) \left( -\frac{e}{m}Ap + \frac{e^2}{2m}A^2 \right) \psi(x)\, dx \tag{302}$$

The last term in $H_I$ can usually be neglected for not too intense fields.

Inserting the expression for the quantized vector potential gives:

$$H_A = \sum_j E_j b_j^+ b_j \tag{303}$$

$$H_I = \hbar \sum_{j,k,\lambda} b_j^+ b_k \left( g_{\lambda jk} a_\lambda + g_{\lambda jk}^* a_\lambda^+ \right) \tag{304}$$

with

$$g_{\lambda jk} = i\frac{e}{m}\sqrt{\frac{1}{2\hbar\omega_\lambda \epsilon_0}} \int \psi_j^*(x)\left(u_\lambda(x) p\right)\psi_k(x)\, dx \tag{305}$$

If $u_\lambda(x)$ varies much more slowly than the extension of the electronic wavefunction ($\lambda_{photon} \gg r_{atom}$, typically $\lambda/r \approx 10^3$) then $u_\lambda(x)$ can be taken out of the integral. In this *electric dipole approximation* one finds

$$\int \psi_j^*(x) p \psi_k(x)\, dx = \frac{im}{\hbar} \int \psi_j^*(x) \left[H_A, x\right] \psi_k(x)\, dx \tag{306}$$

$$= \frac{im}{\hbar}\left(E_j - E_k\right) \int \psi_j^*(x) x \psi_k(x)\, dx \tag{307}$$

$$= im\omega_0 m_{12} \tag{308}$$

Therefore, one can write:

$$H = H_A + H_I + H_{field} \quad \text{with} \tag{309}$$

$$H_A = \sum_j E_j b_j^+ b_j \tag{310}$$

$$H_{field} = \sum_k \hbar\omega_k a_k^+ a_k \tag{311}$$

$$H_I = \hbar \sum_{j,k,\lambda} g_{jk\lambda} b_j^+ b_k (a_\lambda + a_\lambda^+) \tag{312}$$

From the solution of the unperturbed Hamiltonian it can be seen that $b_j^+, b_k$, $a_\lambda, a_\lambda^+$ oscillate rapidly with optical frequencies (e.g. $b(t) = 1/i\hbar\,[H, b]$):

$$b_k = b_k(0)e^{-iE_k t/\hbar} \tag{313}$$

$$b_j^+ = b_j^+(0)e^{iE_j t/\hbar} \tag{314}$$

$$a_\lambda = a_\lambda(0)e^{-i\omega_\lambda t} \tag{315}$$

For not too intense fields only resonant terms with $\omega_0 = (E_j - E_i)/\hbar \simeq \omega_\lambda$ and the form $\exp i(\omega_{ij} - \omega_\lambda)t$ are significant in the dynamics.

In this *rotating wave approximation* the Hamiltonian for the *two-level atom* interacting with a quantized field is:

$$H = H_0 + H_I \tag{316}$$

$$H_0 = \frac{1}{2}\hbar\omega_0\sigma_z + \sum_k \hbar\omega_k a_k^+ a_k \tag{317}$$

$$H_I = \hbar \sum_\lambda g_\lambda (a_\lambda \sigma^+ + a_\lambda^+ \sigma^-) \tag{318}$$

with

$$g_\lambda = -\left(\frac{1}{2\hbar\varepsilon_0\omega_\lambda}\right)^{1/2} \omega_0 u_\lambda(x_0)\mu_{12} \tag{319}$$

with the *dipole moment* $\mu_{12} = em_{12}$

Or if $\omega_\lambda \approx \omega_0$ then

$$g_\lambda = \sqrt{\frac{\omega_0}{2\hbar\varepsilon_0}} u_\lambda(x_0)\mu_{12} \tag{320}$$
$$= \Omega_0/2 \tag{321}$$

with the *vacuum Rabi frequency*

$$\Omega_0 = 2\frac{1}{\hbar}\sqrt{\frac{\hbar\omega_0}{2\varepsilon_0}} u_\lambda(x_0)\mu_{12} = 2\frac{1}{\hbar}\sqrt{\frac{\hbar\omega_0}{2\varepsilon_0 V}} \tilde{u}_\lambda(x_0)\mu_{12} = \frac{2E_0\mu_{12}}{\hbar}\tilde{u}_\lambda(x_0) \tag{322}$$

which is similar as in the classical case, but with the classical field replaced by the electric field per photon and explicitly taking into account the mode function $\tilde{u}_\lambda(x_0)$.

### 7.6 Jaynes-Cummings Model

The most simple case occurs if a single two-level atom interacts with a single mode of the electromagnetic field.

For this case the *Jaynes-Cummings-Hamiltonian* applies:

$$H_{JC} = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega a^+ a + \hbar g(a\sigma^+ + a^+\sigma^-) \tag{323}$$

This Hamiltonian only couples states $|n, e\rangle$ with $|n+1, g\rangle$ where we denote with $|e\rangle, |g\rangle$ the excited and ground state of the two-level atom.

It thus suffices to describe $H$ in this basis and define:

$$H_n = \hbar\left(n + \frac{1}{2}\right)\omega\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \hbar\begin{pmatrix} \delta/2 & g\sqrt{n+1} \\ g\sqrt{n+1} & -\delta/2 \end{pmatrix} \tag{324}$$

with $\delta = \omega_0 - \omega$.

The eigenenergies of this Hamiltonian are:

$$E_{2n} = \hbar\left(n + \frac{1}{2}\right)\omega - \frac{1}{2}\hbar\Omega_n \tag{325}$$
$$E_{1n} = \hbar\left(n + \frac{1}{2}\right)\omega + \frac{1}{2}\hbar\Omega_n \tag{326}$$

with

$$\Omega_n = \sqrt{\delta^2 + \Omega_0^2 (n+1)} \tag{327}$$

$\Omega_n$ is called the generalized Rabi frequency.

The according eigenstates are:

$$|2n\rangle = \cos\vartheta_n |e,n\rangle - \sin\vartheta_n |g,n+1\rangle \tag{328}$$
$$|1n\rangle = \sin\vartheta_n |e,n\rangle + \cos\vartheta_n |g,n+1\rangle \tag{329}$$

with

$$\cos\vartheta_n = \frac{\Omega_n - \delta}{\sqrt{(\Omega_n - \delta)^2 + 4g^2 (n+1)}} \tag{330}$$
$$\sin\vartheta_n = \frac{2g\sqrt{(n+1)}}{\sqrt{(\Omega_n - \delta)^2 + 4g^2 (n+1)}} \tag{331}$$

These eigenstates of the combined atom-field system are called *dressed states*.

On resonance the dressed states reduce to:

$$|2n\rangle = (|e,n\rangle - |g,n+1\rangle)/\sqrt{2} \tag{332}$$
$$|1n\rangle = (|e,n\rangle + |g,n+1\rangle)/\sqrt{2} \tag{333}$$

with eigenenergies:

$$E_{2n} = \hbar\left(n + \frac{1}{2}\right)\omega - \hbar g\sqrt{n+1} \tag{334}$$
$$E_{1n} = \hbar\left(n + \frac{1}{2}\right)\omega + \hbar g\sqrt{n+1} \tag{335}$$

It is easy to show that in the interaction picture (rotating at the frequency $(n+1/2)\omega$) the coefficients $c_{1n}(t), c_{2n}(t)$ of an arbitrary state $|\psi(t)\rangle = c_{1n}(t)|1n\rangle + c_{2n}(t)|2n\rangle$

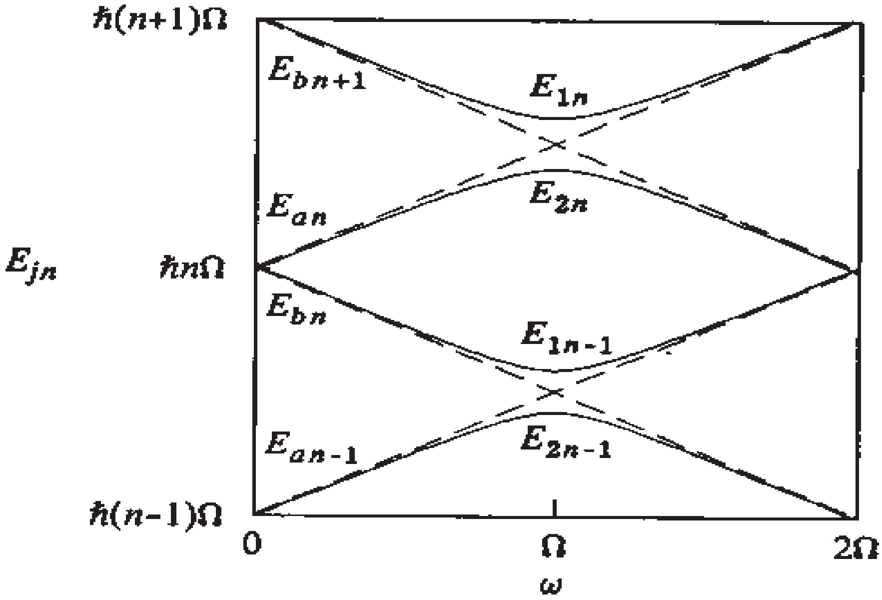


Figure 51: Dressed states. Dashed lines show energy levels without coupling. [from Meystre "Elements of Quantum Optics"]

obey:

$$\begin{pmatrix} c_{2n}(t) \\ c_{1n}(t) \end{pmatrix} = \begin{pmatrix} \exp(i\Omega_n t) & 0 \\ 0 & \exp(-i\Omega_n t) \end{pmatrix} \begin{pmatrix} c_{2n}(0) \\ c_{1n}(0) \end{pmatrix} \tag{336}$$

In the resonant case $\delta = 0$ this gives for a state initially in the upper state:

$$|c_{en}(t)|^2 = \cos^2(g\sqrt{n+1}t) \tag{337}$$

$$|c_{gn+1}(t)|^2 = \sin^2(g\sqrt{n+1}t) \tag{338}$$

Even if $n = 0$ (no photon or interaction with the vacuum) there is:

$$|c_{e0}(t)|^2 = \cos^2(gt) = \frac{1}{2}\left(1 + \cos(\Omega_0 t)\right) \tag{339}$$

Thus there is a coherent exchange of one energy quantum between the atom and the field mode, the so-called *vacuum Rabi oscillation*, in striking difference to the irreversible exponential decay into free space of an excited atom. The periodic energy exchange has an analogy with two coupled pendula.

The coupled equation of motion for the states $|e,n\rangle$ and $|g,n+1\rangle$ are:

$$\dot{c}_{en} = -i\frac{\delta}{2}c_{en} - ig\sqrt{n+1}c_{gn+1} \tag{340}$$

$$\dot{c}_{gn+1} = i\frac{\delta}{2}c_{gn+1} - ig\sqrt{n+1}c_{en} \tag{341}$$

Experiments to show the vacuum Rabi oscillations have been performed recently.

M. Brune, et al., Phys. Rev. Lett. 76, 1800-1803 (1996); B. T. H. Varcoe, S. Brattke, M. Weidinger, H. Walther, Nature 403, 743 - 746 (2000)

### 7.7 Wigner-Weisskopf theory of spontaneous emission

The Hamiltonian for a single two-level atom coupled to a discrete number of modes of an e.magn. field is:

$$H = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\sum_k \omega_k a_k^+ a_k + \hbar\sum_k g_k(a_k\sigma^+ + a_k^+\sigma^-) \tag{342}$$

The most general state vector is:

$$|\psi(t)\rangle = c_{e0}(t)\,|e\{0\}\rangle + \sum_k c_{g\{1_k\}}e^{-i(\omega_k-\omega_0)t}\,|g\{1_k\}\rangle \tag{343}$$

Substituting into the Schroedinger equation gives:

$$\dot{c}_{e0} = -i\sum_k g_k c_{g\{1_k\}}e^{-i(\omega_k-\omega_0)t} \tag{344}$$

$$\dot{c}_{g\{1_k\}} = -ig_k c_{e0}e^{i(\omega_k-\omega_0)t} \tag{345}$$

Formally integrating and inserting results to:

$$\dot{c}_{e0} = -\sum_k g_k^2\int_0^t dt'\; e^{-i(\omega_k-\omega_0)(t-t')}c_{e0}(t') \tag{346}$$

We now move from a discrete set of modes to a continuum by replacing the sum over $k$ with an integral:

$$\sum_k f(k) \longrightarrow \frac{V}{(2\pi)^3}\int d^3k \; f(k) = \frac{V}{(2\pi c)^3}\int d\omega \; \omega^2 \int_0^{\pi} d\vartheta \sin\vartheta \int_0^{2\pi} d\varphi \; f(\omega,\vartheta,\varphi) \quad (347)$$

We also insert

$$g_k^2(\omega,\vartheta) = \frac{1}{\hbar^2}\sum_{\sigma=1}^{2} \left|\langle e|\, e\overrightarrow{r}\widehat{\epsilon}_\sigma \,|g\rangle \, E_{0,\omega}u_\omega\right|^2 \quad (348)$$

$$= \frac{1}{\hbar^2}E_{0,\omega}^2\mu_{12}^2 \sin^2\vartheta \left|\cos^2\varphi + \sin^2\varphi\right| \quad (349)$$

$$= \frac{1}{\hbar^2}E_{0,\omega}^2\mu_{12}^2 \sin^2\vartheta \quad (350)$$

$$= \frac{1}{\hbar^2}\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)\mu_{12}^2 \sin^2\vartheta \quad (351)$$

Inserting and integrating gives:

$$\dot{c}_{e0} = -\frac{1}{6\varepsilon_0\pi^2\hbar c^3}\int d\omega \; \omega^3\mu_{12}\int_0^t dt' \; e^{-i(\omega_k-\omega_0)(t-t')}c_{e0}(t') \quad (352)$$

with

$$\lim_{t\to\infty}\int_0^t dt' \; e^{-i(\omega_k-\omega_0)(t-t')} = \pi\delta(\omega-\omega_0) - P\left[\frac{i}{\omega-\omega_0}\right] \quad (353)$$

it follows:

$$\dot{c}_{e0} = -\frac{\Gamma}{2}c_{e0}(t) \quad (354)$$

where the Lamb-shift is neglected.

The rate $\Gamma$ is the Wigner-Weisskopf rate of spontaneous emission:

$$\Gamma = \frac{\omega^3\mu_{12}^2}{3\pi\varepsilon_0\hbar c^3} \quad (355)$$

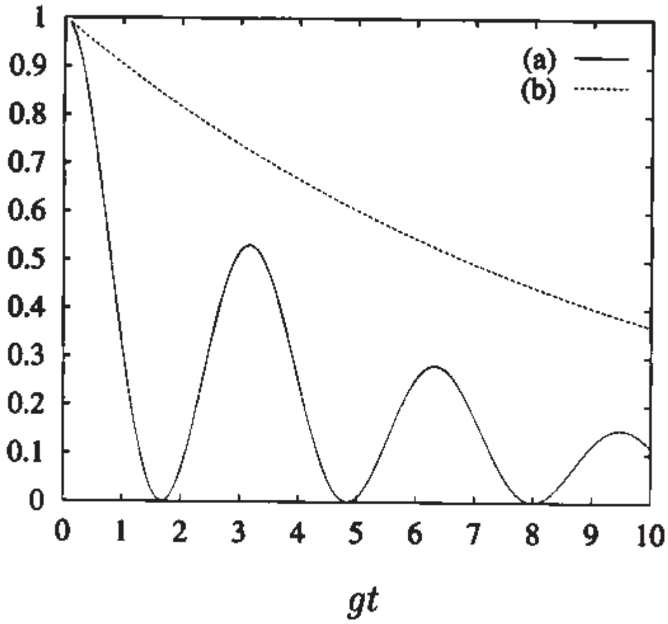


Figure 52: Probability to find an excited atom in a cavity in the upper state for weak damping (a) and strong damping (b) of the cavity field. [from Scully "Quantum optics"]

## 7.8 Collapse and Revival & Quantum beats

### 7.8.1 Collapse & Revival

An interesting phenomenon exists if a single atom interacts not with a single Fock-state $|n\rangle$, but with a coherent state $|\alpha\rangle$ where

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{356}$$

In this case the probability to find the initially excited atom in the excited state after some time $t$ is:

$$P_e = \sum_n p_n \left|c_{en}(t)\right|^2 \tag{357}$$

$$= e^{-|\alpha|^2} \sum_n \frac{|\alpha|^{2n}}{n!} \cos^2(g\sqrt{n+1}t) \tag{358}$$

The time evolution is a sum of oscillations with different Rabi frequencies which then dephase.
This occurs on a timescale of appr.:

$$t_c \approx g^{-1} \tag{359}$$

However, after some time there is a *revival* of the probability to find the atom excited again. This is a pure quantum effect and due to the discrete number of basis states of the coherent state.

The time for the revival can be estimated to:

$$t_r \approx 4\pi\sqrt{\bar{n}}t_c = 4\pi\sqrt{\bar{n}}g^{-1} \tag{360}$$

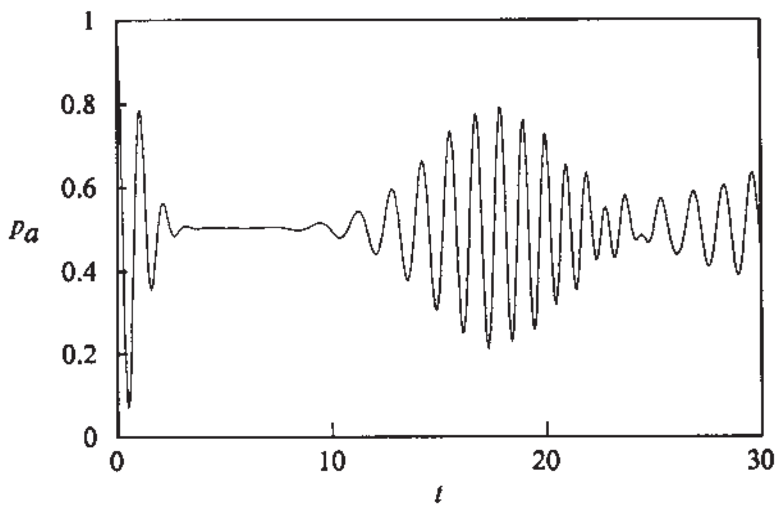


Figure 53: Collapse and revival as measured by G. Rempe, H. Walther, and N. Klein, Phys. Rev. Lett. 58, 353-356 (1987) [from Meystre "Elements of Quantum optics"]

#### 7.8.2 Quantum Beats

Another interesting quantum effect in the spontaneous emission of light from a single atom is the quantum beat effect.

Consider the following $\Lambda$- and $V$-type three level systems:

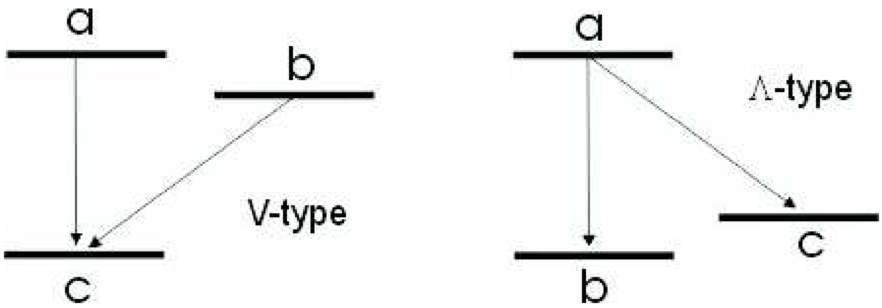


Figure 54: V-type and Λ-type three level systems

Assume the atomic state is in a superposition:

$$|\psi(t)\rangle = c_a e^{-i\omega_a t} |a\rangle + c_b e^{-i\omega_b t} |b\rangle + c_c e^{-i\omega_c t} |c\rangle \tag{361}$$

Semiclassically there exist oscillating dipoles:

| | | | |
|---|---|---|---|
| $V$-type system: | $P_{ac}$ | and | $P_{bc}$ |
| $\Lambda$-type system: | $P_{ab}$ | and | $P_{ac}$ |

which create a field of the form

$$E(t) = E_{01}\exp(-i\nu_1 t) + E_{02}\exp(-i\nu_2 t) \tag{362}$$

with

$$\begin{array}{ll} V\text{-type system:} & \nu_1 = \omega_a - \omega_c \\ & \nu_2 = \omega_b - \omega_c \\ \Lambda\text{-type system:} & \nu_1 = \omega_a - \omega_b \\ & \nu_2 = \omega_a - \omega_c \end{array}$$

Obviously this creates a beating in a square law detector which can only measure intensities:

$$|E(t)|^2 = |E_{01}|^2 + |E_{02}|^2 + \{E_{01}^* E_{02} \exp[i(\nu_1 - \nu_2)t] + c.c.\} \tag{363}$$

for both the $\Lambda$- and the $V$-type system.

However, in the quantum case the beating signal is given by the following expressions:

$V$-type system:

$$I = \langle \psi_V(t)| E_1^{(-)} E_2^{(+)} |\psi_V(t)\rangle \tag{364}$$

with

$$E_1^{(-)} \propto a_1^+ e^{i\nu_1 t} \quad \text{and} \quad E_1^{(+)} \propto a_2 e^{-i\nu_2 t} \tag{365}$$

Therefore with the state

$$|\psi_V(t)\rangle = \sum_{i=a,b,c} c_i |i, 0\rangle + c_1 |c, 1_{\nu 1}\rangle + c_2 |c, 1_{\nu 2}\rangle \tag{366}$$

this gives:

$$\begin{aligned} I &= const. \langle 1_{\nu 1} 0_{\nu 2}| a_1^+ a_2 |0_{\nu 1} 1_{\nu 2}\rangle \exp[i(\nu_1 - \nu_2)t] \langle c|c\rangle && (367) \\ &= const. \langle 1_{\nu 1} 0_{\nu 2}| a_1^+ a_2 |0_{\nu 1} 1_{\nu 2}\rangle \exp[i(\nu_1 - \nu_2)t] && (368) \end{aligned}$$

But, in the $\Lambda$-type system:

$$|\psi_\Lambda(t)\rangle = \sum_{i=a,b,c} c_i' |i, 0\rangle + c_1' |b, 1_{\nu 1}\rangle + c_2' |c, 1_{\nu 2}\rangle \tag{369}$$

and

$$\begin{aligned} I &= \langle \psi_\Lambda(t)| E_1^{(-)} E_2^{(+)} |\psi_\Lambda(t)\rangle && (370) \\ &= const. \langle 1_{\nu 1} 0_{\nu 2}| a_1^+ a_2 |0_{\nu 1} 1_{\nu 2}\rangle \exp[i(\nu_1 - \nu_2)t] \langle c|b\rangle && (371) \\ &= 0 && (372) \end{aligned}$$

There is no beat note in the $\Lambda$-system.

This can be interpreted in the framework of which-path-information:

- After emission of a photon it is not possible to say which decay path the photon took in the $V$-system. Thus, the two paths interfere.
- In the $\Lambda$-system a measurement of the atomic state (it is either in $|b\rangle$ or $|c\rangle$) reveals information which path the photon took (even before it is detected). thus there is no interference.

This effect has some analogy to Young's double slit experiment.