8 System Reservoir Interactions

8.1 The master equation

In quantum optics incoherent and irreversible processes can be described by the master equation.

To define the problem we have a 'small' system S which interacts with a 'large' system R beeing a reservoir (here we consider only thermal baths, i.e. reservoirs in thermal equilibrium).

The Hamiltonian for such a model system is:

$$H = H_S + H_R + V \tag{373}$$

The equation of motion for the total density operator $\rho^{(T)}$ (system and reservoir) in the interaction picture is:

$$\frac{\partial \rho^{(T)}}{\partial t} = -\frac{i}{\hbar} \left[V(t), \rho^{(T)} \right] \tag{374}$$

The reduced density operator (of the system) is obtained by tracing over the reservoir variables:

$$\rho(t) = Tr_R \left\{ \rho^{(T)} \right\} \tag{375}$$

Initially we assume:

$$\rho^{(T)}(0) = \rho(0) \otimes \rho_R \tag{376}$$

Formal integration and iteration leads to:

$$\rho^{(T)}(t) = \rho^{(T)}(0) + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar} \right)^n \int_0^t dt_1 \int_0^{t_1} dt_2 ... \tag{377}$$

$$\times \int_0^{t_{n-1}} dt_n \left[V(t_1), \left[V(t_2), ... \left[V(t_n), \rho^{(T)}(0) \right] \right] \right]$$

and

$$\rho(t) = \rho(0) + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar}\right)^n \int_0^t dt_1 \int_0^{t_1} dt_2 ... \tag{378}$$

$$\times \int_0^{t_{n-1}} dt_n \, Tr_R \{[V(t_1), [V(t_2), ... [V(t_n), \rho_R \otimes \rho(0)]]]\}$$

$$= (1 + U_1(t) + U_2(t) + ...) \rho(0)$$

$$\doteq U(t)\rho(0)$$

where

$$U_n(t) = \left(-\frac{i}{\hbar}\right)^n \int_0^t dt_1 \int_0^{t_1} dt_2 ... \tag{379}$$

$$\times \int_0^{t_{n-1}} dt_n \, Tr_R \{[V(t_1), [V(t_2), ... [V(t_n), \rho_R \otimes (\cdot)]]]\} \tag{380}$$

It is possible to define

$$\frac{\partial \rho(t)}{\partial t} = L(t)\rho(t) = \left[\dot{U}_1(t) + \dot{U}_2(t) + ...\right] U(t)^{-1}\rho(t) \tag{381}$$

If we assume for $V(t)$:

$$Tr_R (V(t)\rho_R) = 0 \tag{382}$$

it follows $U_1(t) = 0$. Keeping only terms of second order it follows:

$$L(t) \approx \dot{U}_2(t) = -\frac{1}{\hbar^2} \int_0^t dt_1 \, Tr_R \{[V(t), [V(t_1), \rho_R \otimes (\cdot)]]\} \tag{383}$$

$$\frac{\partial \rho(t)}{\partial t} = L(t)\rho(t) \approx -\frac{1}{\hbar^2} \int_0^t dt_1 \, Tr_R \{[V(t), [V(t_1), \rho_R \otimes \rho(t)]]\}$$

Now we assume as a special case an harmonic oscillator as system S damped by coupling to a thermal reservoir R of harmonic oscillators.

Thus:

$$V(t) = \hbar \left(a^+ \Gamma(t) e^{i\omega_0 t} + a \Gamma^+(t) e^{-i\omega_0 t} \right) \quad \text{with} \tag{384}$$

$$\Gamma(t) = \sum_j g_j b_j e^{-i\omega_j t} \quad \text{where} \tag{385}$$

$$\left[b_j, b_k^+ \right] = \delta_{jk} \tag{386}$$

The b_j denote the bosonic harmonic oscillators in the reservoir!

Substituting gives terms with the following integrals:

$$I_1 = \int_0^t dt_1 \sum_{i,j} g_i g_j \left\langle b_i b_j \right\rangle_R e^{-i(\omega_i t + \omega_j t_1)} e^{i\omega_0(t+t_1)} \tag{387}$$

$$\approx \int_0^t dt_1 \int_0^\infty \frac{d\omega_1}{2\pi} D(\omega_1) \int_0^\infty \frac{d\omega_2}{2\pi} D(\omega_2) g(\omega_1) g(\omega_2) \left\langle b_i b_j \right\rangle_R e^{-i(\omega_i t + \omega_j t_1) + i\omega_0(t+t_1)}$$

$$I_2 = \int_0^t dt_1 \sum_{i,j} g_i^* g_j^* \left\langle b_i^+ b_j^+ \right\rangle_R e^{i(\omega_i t + \omega_j t_1)} e^{-i\omega_0(t+t_1)} \tag{388}$$

$$\approx \int_0^t dt_1 \int_0^\infty \frac{d\omega_1}{2\pi} D(\omega_1) \int_0^\infty \frac{d\omega_2}{2\pi} D(\omega_2) g^*(\omega_1) g^*(\omega_2) \left\langle b_i^+ b_j^+ \right\rangle_R e^{i(\omega_i t + \omega_j t_1) - i\omega_0(t+t_1)}$$

$$I_3 = \int_0^t dt_1 \sum_{i,j} g_i g_j^* \left\langle b_i b_j^+ \right\rangle_R e^{-i(\omega_i t - \omega_j t_1)} e^{i\omega_0(t-t_1)} \tag{389}$$

$$\approx \int_0^t dt_1 \int_0^\infty \frac{d\omega_1}{2\pi} D(\omega_1) \int_0^\infty \frac{d\omega_2}{2\pi} D(\omega_2) g(\omega_1) g^*(\omega_2) \left\langle b_i b_j^+ \right\rangle_R e^{-i(\omega_i t - \omega_j t_1) + i\omega_0(t-t_1)}$$

$$I_4 = \int_0^t dt_1 \sum_{i,j} g_i g_j \left\langle b_i^+ b_j \right\rangle_R e^{i(\omega_i t - \omega_j t_1)} e^{-i\omega_0(t-t_1)} \tag{390}$$

$$\approx \int_0^t dt_1 \int_0^\infty \frac{d\omega_1}{2\pi} D(\omega_1) \int_0^\infty \frac{d\omega_2}{2\pi} D(\omega_2) g^*(\omega_1) g(\omega_2) \left\langle b_i^+ b_j \right\rangle_R e^{i(\omega_i t - \omega_j t_1) - i\omega_0(t-t_1)}$$

where sums were replaced by integrals and $D(\omega_j)$ denotes the density of states.

If we assume a thermal bath with *no* phase dependent correlations ($\langle b_i b_j \rangle_R = \langle b_i^+ b_j^+ \rangle_R = 0$) then only the integrals I_3 and I_4 are non-zero.

We assume δ-correlations:

$$\langle b_i^+(\omega_1) b_j(\omega_2) \rangle_R = 2\pi \overline{n}_R(\omega_1) \delta(\omega_1 - \omega_2) \tag{391}$$

$$\langle b_i(\omega_1) b_j^+(\omega_2) \rangle_R = 2\pi (\overline{n}_R(\omega_1) + 1) \delta(\omega_1 - \omega_2) \tag{392}$$

where $\overline{n}_R(\omega_1)$ is the mean number of photons of the reservoir at frequency ω_1.

Inserting, extending the time integration to ∞ (the integrals over the frequency vanish rapidly), and defining $\epsilon = \omega - \omega_0$ and $\tau = t - t_1$ gives e.g. for I_3 :

$$I_3 \approx \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} D^2(\epsilon + \omega_0) g^2(\varepsilon + \omega_0)(\overline{n}_R(\varepsilon + \omega_0) + 1) \int_0^{\infty} d\tau \ e^{-i\epsilon\tau} \tag{393}$$

We note that

$$\int_0^{\infty} d\tau \ e^{-i\epsilon\tau} = \pi\delta(\epsilon) - iPV\left(\frac{1}{\epsilon}\right) \tag{394}$$

Now we can write

$$I_3 = \frac{\gamma}{2}(\overline{n}_R(\omega_0) + 1) - i\Delta \quad \text{and similar} \tag{395}$$

$$I_4 = \frac{\gamma}{2}\overline{n}_R(\omega_0) - i\Delta' \quad \text{with} \tag{396}$$

$$\gamma = D^2(\omega_0) g^2(\omega_0) \tag{397}$$

Substituting the integrals, omitting the phase terms Δ (which cause an energy shift) and inculding an additional time evolution (according to H_I) of the system S finally gives the *master equation*:

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H_0 + H_I, \rho] \tag{398}$$

$$+ \frac{\gamma}{2}(\overline{n}_R + 1)(2a\rho a^+ - a^+ a\rho - \rho a^+ a)$$

$$+ \frac{\gamma}{2}(\overline{n}_R)(2a^+ \rho a - aa^+ \rho - \rho aa^+)$$

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As an example consider for $H_I = 0$ the time evolution of the mean photon number $n = \langle a^+ a \rangle$ of the system:

$$\frac{d \langle a^+ a \rangle}{dt} = -\gamma \langle a^+ a \rangle + \gamma \bar{n}_R \tag{399}$$

and thus

$$n(t) = n_0 e^{-\gamma t} + \bar{n}_R \left(1 - e^{-\gamma t}\right) \tag{400}$$

8.2   Representations of the master equation

8.2.1   Fock representation

It is easy to represent the master equation in a Fock basis:

$$\frac{d}{dt} \rho_{mn} = \gamma \bar{n}_R \left[(nm)^{1/2} \rho_{m-1n-1} - \frac{1}{2}(m+n+2)\rho_{mn} \right] \tag{401}$$
$$+ \gamma(\bar{n}_R + 1) \left[\{(n+1)(m+1)\}^{1/2} \rho_{m+1n+1} - \frac{1}{2}(m+n)\rho_{mn} \right]$$

For the diagonal elements which describe the probability p_n to find exactly n photons in the mode it follows:

$$\frac{d}{dt} p_n = \gamma \bar{n}_R n p_{n-1} + \gamma(\bar{n}_R + 1)(n+1)p_{n+1} \tag{402}$$
$$- \left[\gamma \bar{n}_R(n+1) + \gamma(\bar{n}_R + 1)n \right] p_n$$

In steady state the *detailed balance condition* holds:

$$\gamma(\bar{n}_R + 1)n p_n = \gamma \bar{n}_R n p_{n-1} \tag{403}$$

This leads to the thermal probability disribution:

$$\bar{p}_n = \frac{1}{1 + n_R} \left(\frac{n_R}{1 + n_R} \right)^n \tag{404}$$

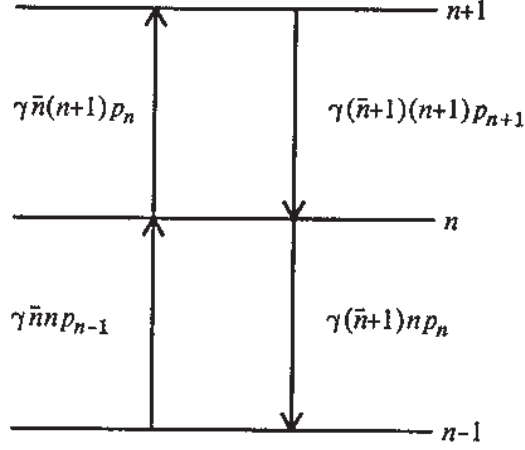The following picture gives a schematis of the probability flow between the p_n:

Figure 55: Schematis of the probability flow between different diagonal matrix elements [from Meystre "Elements of Quantum Optics"]

8.2.2 Fokker-Planck-equation

A P-representation transforms the master equation into a statistical c-number equation if the P-distribution is interpreted as a (quasi-) probability distribution.

One can show that:

$$\frac{\partial}{\partial t}P(\alpha) = \left[\frac{1}{2}\gamma\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^*\right) + \gamma n_R\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]P(\alpha) \tag{405}$$

This equation is a *Fokker-Planck equation.*

The first term

$$\frac{1}{2}\gamma\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^*\right) \tag{406}$$

is called the drift term and the second term

$$\gamma n_R\frac{\partial^2}{\partial\alpha\partial\alpha^*} \tag{407}$$
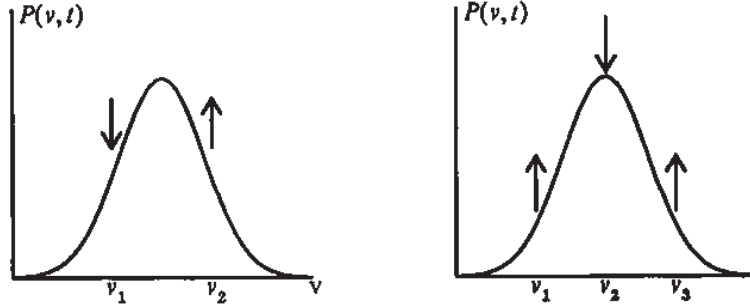
is called the diffusion term.

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Figure 56: Qualitative effect of the drift term (left) and the diffusion term (right) on a probability distribution in the Fokker-Planck equation [from Meystre "Elements of Quantum Optics"]

The drift term describes the evolution of the average values, e.g. $\langle \alpha \rangle_P$:

$$\frac{\partial}{\partial t} \langle \alpha \rangle_P = -\frac{\gamma}{2} \langle \alpha \rangle_P \tag{408}$$

or $\langle \alpha^* \alpha \rangle_P$:

$$\frac{\partial}{\partial t} \langle \alpha^* \alpha \rangle_P = -\gamma \langle \alpha^* \alpha \rangle_P + \gamma n_R \tag{409}$$

The diffusion term describes the evolution of higher moments, e.g. of the width of the quasi-probability distribution.

8.3 Quantum Langevin equation

The previous description of a small system interacting with a large reservoir was performed in the Schroedinger picture.

Additional insight can be gained in the Heisenberg picture.

Starting again with the Hamiltonian:

$$H = \hbar\omega_0 a^+ a + \sum_j \hbar\omega_j b_j^+ b_j + \hbar\sum_j \left(g_j a^+ b_j + g_j^* a b_j^+ \right) \tag{410}$$

gives:

$$\dot{a}(t) = -i\omega_0 a(t) - i\sum_j g_j b_j(t) \tag{411}$$

$$\dot{b_j}(t) = -i\omega_j b_j(t) - ig_j^* a(t) \tag{412}$$

Formally integrating results in

$$b_j(t) = b_j(t_0)e^{-i\omega_j(t-t_0)} - ig_j^* \int_{t_0}^{t} dt' \ a(t')e^{-i\omega_j(t-t')} \tag{413}$$

$$= b_{free}(t) + b_{radiation}(t) \tag{414}$$

In this definition $b_{radiation}(t)$ describes the modification of the free evolution due to the system-reservoir interaction.

Inserting gives:

$$\dot{a}(t) = -i\omega_0 a(t) - i\sum_j g_j b_j(t_0)e^{-i\omega_j(t-t_0)} - \sum_j |g_j|^2 \int_{t_0}^{t} dt' \ a(t')e^{-i\omega_j(t-t')} \tag{415}$$

Transforming in the interaction picture with

$$A(t) = a(t)e^{i\omega_0 t} \tag{416}$$

finally gives:

$$\dot{A}(t) = -\sum_j |g_j|^2 \int_{t_0}^{t} dt' \ A(t')e^{-i(\omega_j-\omega_0)(t-t')} + F(t) \tag{417}$$

$$\text{with} \quad F(t) = -i\sum_j g_j b_j(t_0)e^{-i(\omega_j-\omega_0)(t-t_0)} \tag{418}$$

The integral over t' above is very similar to the integral which occured in the Wigner-Weisskopf theory of spontaneous emission.

Thus, one can evaluate the integrals and finds:

$$\dot{A}(t) = -\frac{\gamma}{2}A(t) + F(t) \tag{419}$$

$$\text{with} \quad \langle F(t)\rangle_R = 0 \tag{420}$$

The noise operator $F(t)$ plays a similar role as in a classical Langevin equation. Therefore, the equation above is called the *quantum Langevin equation.*

Remarks:

- One could not expect to find a simple time evolution for the operators $A(t)$ such as $A(t) = A(t_0) \exp(-t\gamma/2)$. This would imply a possible change of the commutation relation with time in striking contradiction to the laws of quantum mechanics.

- It is, however, $\left\langle \dot{A}(t) \right\rangle = -\frac{\gamma}{2} \left\langle A(t) \right\rangle$

- The Heisenberg picture has the appealing advantage that the equations look similar as in the classical case.

- The operator ordering is very important when caculating in the Heisenberg picture. System and reservoir operators may commute at time zero, but not necessarily at a later time. The ordering can be chosen at the beginning, but then has to be used consequently.

8.4 Resonance fluorescence

8.4.1 Master equation

Resonance fluorescence is the light that is detected from an atom that is driven by a laser field. Here it is assumed that the laser field is strong and can be described classically.

The system may be described by the following Hamiltonian:

$$H = H_0 + H_{drive} + H_{damping} \tag{421}$$

$$H_0 = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\sum \omega_k b_k^+ b_k \tag{422}$$

$$H_{drive} = \hbar\left(g\frac{\varepsilon^*}{2}\sigma^- e^{i\omega t} + g^*\frac{\varepsilon}{2}\sigma^+ e^{-i\omega t} \right) \tag{423}$$

$$H_{damping} = \hbar\sum \left(g_k b_k^+ \sigma^- + g_k^* b_k \sigma^+ \right) \tag{424}$$

where ε is the amplitude of the classical driving field.

The master equation for the reduced density operator for the system (the atom) is if we assume the reservoir at zero temperature:

$$\frac{\partial \rho}{\partial t} = \left(\frac{\partial \rho}{\partial t}\right)_{coherent} + \left(\frac{\partial \rho}{\partial t}\right)_{incoherent} \tag{425}$$

In the interaction picture (with the detuning $\Delta \omega$):

$$\frac{\partial \rho}{\partial t} = -i\frac{\Delta \omega}{2} \left[\sigma_z, \rho\right] - i\frac{\Omega}{2} \left[\sigma^+ + \sigma^-, \rho\right] + \tag{426}$$
$$\frac{\gamma}{2} \left(2\sigma^- \rho \sigma^+ - \rho \sigma^+ \sigma^- - \sigma^+ \sigma^- \rho\right)$$

where γ is the natural linewidth of the atom and $\Omega = \varepsilon g^*/\hbar$.

From the master equation the equations of motion for the expectation values $\langle \sigma^-(t)\rangle$, $\langle \sigma^+(t)\rangle$ and $\langle \sigma_z(t)\rangle$ can be derived:

$$\frac{d}{dt} \langle \sigma^+ \rangle = -\left(\frac{\gamma}{2} - i\Delta \omega\right) \langle \sigma^+ \rangle - i\Omega \langle \sigma_z \rangle \tag{427}$$

$$\frac{d}{dt} \langle \sigma^- \rangle = -\left(\frac{\gamma}{2} + i\Delta \omega\right) \langle \sigma^- \rangle + i\Omega \langle \sigma_z \rangle \tag{428}$$

$$\frac{d}{dt} \langle \sigma_z \rangle = -\gamma \left(1 + \langle \sigma_z \rangle\right) - i\Omega \left(\langle \sigma^+ \rangle - \langle \sigma^- \rangle\right) \tag{429}$$

In these equation we recognize the optical Bloch equations for a two-level system driven by a single-mode electromagnetic field.

The steady state solution in the interaction picture of this equation gives:

$$\langle \sigma_z \rangle_{ss} = -\frac{1 + (2\Delta \omega/\gamma)^2}{1 + (2\Delta \omega/\gamma)^2 + 2(\Omega/\gamma)^2} \tag{430}$$

$$\langle \sigma^+ \rangle_{ss} = -i\frac{(\Omega/\gamma)(1 + i2\Delta \omega/\gamma)}{1 + (2\Delta \omega/\gamma)^2 + 2(\Omega/\gamma)^2} \tag{431}$$

In the interaction picture the exact solution of the Bloch equation at zero detuning $(\Delta \omega = 0)$ is:

$$\langle \sigma_z(t)\rangle = \frac{2\Omega^2}{\gamma^2 + 2\Omega^2} \left[1 - e^{-3\gamma t/4}\left(\cosh \kappa t + \frac{3\gamma}{4\kappa} \sinh \kappa t\right)\right] - 1 \tag{432}$$

$$\langle \sigma^+(t)\rangle = i\Omega \frac{\gamma}{\gamma^2 + 2\Omega^2} \left\{1 - e^{-3\gamma t/4}\left[\cosh \kappa t + \left(\frac{\kappa}{\gamma} + \frac{3\gamma}{16\kappa}\right) \sinh \kappa t\right]\right\} \tag{433}$$

$$\langle \sigma^-(t)\rangle = \langle \sigma^+ \rangle^* \tag{434}$$

where $\kappa = \left[\left(\frac{\gamma}{4}\right)^2 - \Omega^2 \right]^{1/2}$

There is a threshold at $\Omega = \gamma/4$ below which the solutions are monotonic and above which they exhibit oscillations. For the case $\Omega \gg \gamma/4$ the solutions are:

$$\langle \sigma_z(t) \rangle = -e^{-3\gamma t/4} \cos \Omega t \tag{435}$$

$$\langle \sigma^+(t) \rangle = \frac{i}{2} e^{-3\gamma t/4} \sin \Omega t \tag{436}$$

The probability for the atom to be in the excited state is:

$$P_e(t) = \frac{1}{2} \left(1 - e^{-3\gamma t/4} \cos \Omega t \right) \tag{437}$$

The steady state probability is:

$$P_e^{(ss)}(t) = \frac{\Omega^2}{\gamma^2 + 2\Omega^2} \tag{438}$$

This equation shows that for intense fields the atom does not respond to the driving field like a classical oscillator, but saturates at a level:

$$\lim_{\Omega \to \infty} P_e^{(ss)}(t) = \frac{1}{2} \tag{439}$$

8.4.2 Spectrum of the fluorescent light

The spectrum of the fluorescent light is given by the following expression:

$$S(\omega) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{-i\omega\tau} \left\langle E^{(-)}(t) E^{(+)}(t+\tau) \right\rangle_{ss} d\tau \tag{440}$$

It is possible to relate the electric field emitted by the atom directly to the operator σ^-:

$$\left\langle E^{(+)}(r,t) \right\rangle = -U(r) \left\langle \sigma^-(t) \right\rangle \tag{441}$$

Hier $U(r)$ is an amplitude function. (Here we write t, but we mean the retarded time $t - r/c$!)

The steady state spectrum may thus be written as an integral over the correlations of σ^-:

$$S(\omega) = \frac{I_0(r)}{2\pi} \int\limits_{-\infty}^{\infty} e^{-i\omega\tau} \left\langle \sigma^+(t+\tau)\sigma^-(t) \right\rangle_{ss} d\tau \quad \text{where } I(r) = |U(r)|^2 \tag{442}$$

In order to find the integral we have to derive the time evolution of the two-time correlation function $\left\langle \sigma^+(t+\tau)\sigma^-(t) \right\rangle_{ss}$. There is a very helpful theorem, the *quantum regression theorem*, which states the following: If $\left\langle \sigma^+(\tau) \right\rangle$ (time evolution) obeys a certain linear differential equation, then the time evolution of a two-time correlation function $\left\langle \sigma^+(t+\tau)\sigma^-(t) \right\rangle$ obeys exactly the same differential equation.

The initial condition is found by the commutation relation of the Pauli matrices:

$$\left\langle \sigma^+(0)\sigma^-(0) \right\rangle_{ss} = \frac{1}{2} \left(\left\langle \sigma^z \right\rangle_{ss} + 1 \right) \tag{443}$$

Thus, it is straightforward to find the time evolution for $G^{(1)}(\tau) = \left\langle \sigma^+(\tau)\sigma^-(0) \right\rangle_{ss}$ from the equations above.

After some calculation one finds:

$$G^{(1)}(\tau) = I_0(r)\frac{\Omega^2}{\gamma^2 + 2\Omega^2}\left(\frac{\gamma^2}{\gamma^2 + 2\Omega^2}e^{-i\omega_0\tau} + \exp\left[-\left(\frac{\gamma}{2} + i\omega_0\right)\tau \right] \right. \tag{444}$$

$$- \lambda_+ \exp\left\{ -\left[\left(\frac{3\gamma}{4} - \kappa\right) + i\omega_0 \right]\tau \right\}$$

$$\left. + \lambda_- \exp\left\{ -\left[\left(\frac{3\gamma}{4} + \kappa\right) + i\omega_0 \right]\tau \right\} \right)$$

where

$$\lambda_\pm = \frac{1}{4}\left(\frac{-\frac{\gamma}{4\kappa}(10\Omega^2 - \gamma^2) \pm (\gamma^2 - 2\Omega^2)}{\gamma^2 + 2\Omega^2} \right) \tag{445}$$

This rather complicated result requires some discussion:
The spectrum will result from a Fourier transform of $G^{(1)}(\tau)$. The first term corresponds to elastic scattering, whereas the second term corresponds to ineleastic scattering by a Lorentzian peaked at ω_0. The third and fourth terms contribute to this term if $\Omega < \gamma/4$, but if $\Omega > \gamma/4$ they give rise to two side bands of width $3\gamma/4$.

Thus at low intensities the spectrum is

$$S(\omega) = I_0(r)\frac{\gamma^2}{\gamma^2 + 2\Omega^2}\delta(\omega - \omega_0) \tag{446}$$
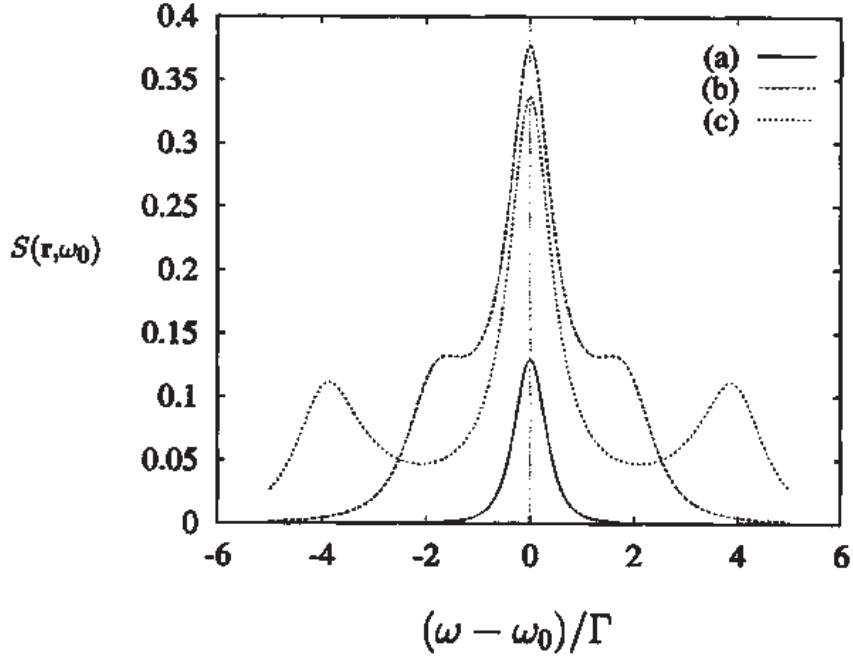
Figure 57: Calculated fluorescence spectrum of a single driven two-level atom (Mollow triplet) for increasing (from a) to c)) pump power. [from Scully "Quantum Optics"]

If the driving power is increased the peak broadens until when $\Omega > \gamma/4$ two side peaks appear.

The spectrum for $\Omega \gg \gamma/4$ is:

$$S(\omega) = \frac{I_0(r)}{2\pi}(\frac{\pi\gamma^2}{\gamma^2 + 2\Omega^2}\delta(\omega - \omega_0) + \frac{1}{2}\frac{\frac{1}{2}\gamma}{\left(\frac{1}{2}\gamma\right)^2 + (\omega - \omega_0)^2} \qquad (447)$$

$$+ \frac{1}{4}\frac{\frac{3}{4}\gamma}{\left(\frac{3}{4}\gamma\right)^2 + (\omega - \omega_0 + \Omega)^2} + \frac{1}{4}\frac{\frac{3}{4}\gamma}{\left(\frac{3}{4}\gamma\right)^2 + (\omega - \omega_0 + \Omega)^2})$$

For very large driving fields the elastic scattering is negligible. The spectrum consists of three Lorentzians, the *Mollow triplet*.

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This spectrum has a very intuitive explanation in the picture of dressed states. The three peaks results simply from the spontaneous emission between different dressed states.
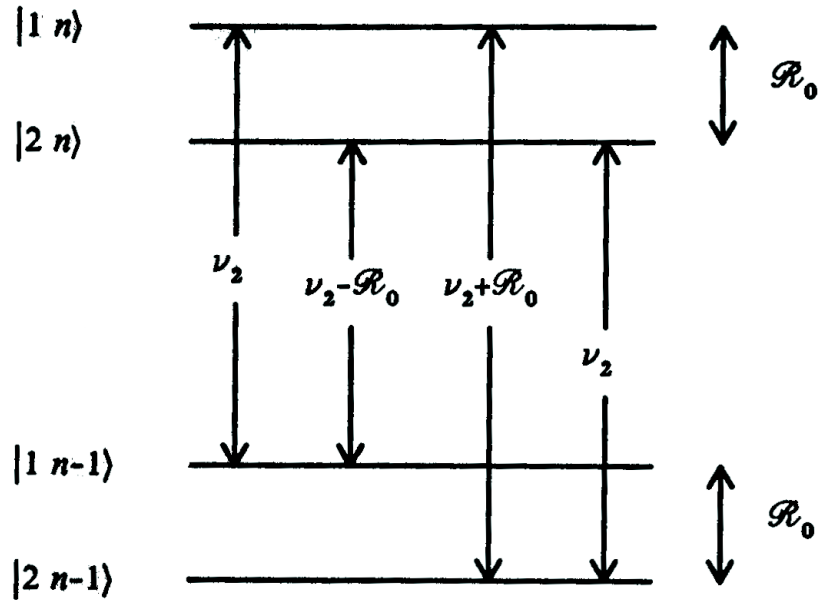

Figure 58: Interpretation of the resonance fluorescence spectrum as spontaneous decay betwen dressed state energy levels [from Meystre "Elements of Quantum Optics"]

First experiments to measure this spectrum were performed by Ezekiel (Wu et al., Phys. Rev. Lett. 35, 1426 (1975)) and Walther (Diedrich and Walther, Phys. Rev. Lett. 58, 203 (1987)).
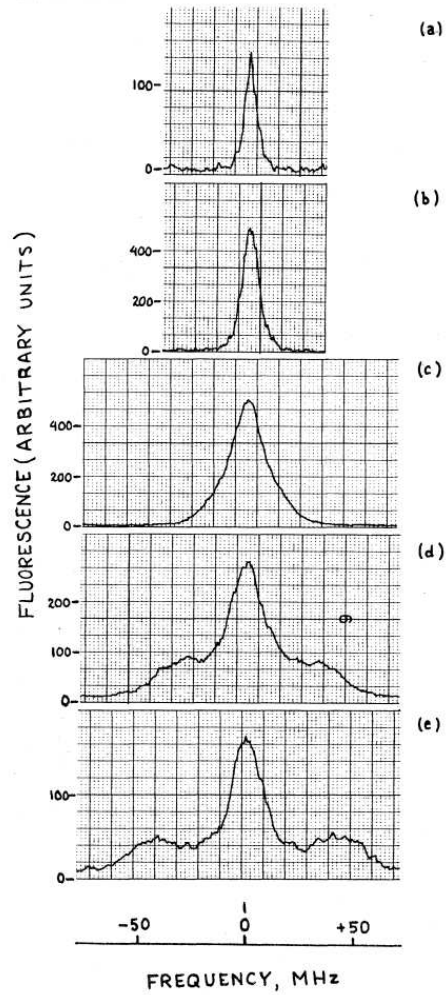
Figure 59: Measured resonance fluorescence from a single Na atom in an atomic beam at 589 nm for the driving powers 0,8 mW/cm^2(a), 8 mW/cm^2(b), 85 mW/cm^2(c), 490 mW/cm^2(d), 920 mW/cm^2(e) [Wu et al., 1975]

8.4.3 Photon correlation

In addition to the first order coherence function $G^{(1)}(\tau)$ the second order coherence function $G^{(2)}(\tau)$ can be evaluated.

It is

$$G^{(2)}(r, t; r, t + \tau) = \left\langle E^{(-)}(r, t) E^{(-)}(r, t + \tau) E^{(+)}(r, t + \tau) E^{(+)}(r, t) \right\rangle \qquad (448)$$

This may be expressed in terms of the atomic operators similar as in the previous subsection:

$$G^{(2)}(r, t; r, t + \tau) = I_0^2(r) \left\langle \sigma^+(t) \sigma^+(t + \tau) \sigma^-(t + \tau) \sigma^-(t) \right\rangle \qquad (449)$$

With the relation $\sigma^+ \sigma^- = \frac{1}{2}(\sigma_z + 1)$ this can be written as:

$$\left\langle \sigma^+(t) \sigma^+(t + \tau) \sigma^-(t + \tau) \sigma^-(t) \right\rangle = 1/2 \left\langle \sigma^+(t)(\sigma_z(t) + 1) \sigma^-(t) \right\rangle \qquad (450)$$

The solution can be found using the quantum regression theorem:

$$g^{(2)}(\tau) = 1 - e^{-3\gamma t/4} \left(\cosh \kappa\tau + \frac{3\gamma}{4\kappa} \sinh \kappa\tau \right) \qquad (451)$$

This has the value $g^{(2)}(0) = 0$ and is thus anti-bunched light.

Similar as in the case of the spectrum the behaviour of $g^{(2)}(\tau)$ changes when the driving field is increased.

For low power ($\Omega < \gamma/4$) there is a monotonic increase:

$$g^{(2)}(\tau) = \left(1 - e^{-\gamma t/2} \right)^2 \qquad (452)$$

whereas for high power ($\Omega \gg \gamma/4$) there is an oscillatory behaviour:

$$g^{(2)}(\tau) = 1 - e^{-3\gamma t/4} \cos \Omega t \qquad (453)$$

The interpretation of this result is obvious when the correlation function is discussed in terms of photon correlation:
When a photon from a single atom is deteced, then the atom is projected to its ground state. The probability to detect the next photon is then proportional to the probability to find the atom in its excited state again, i.e., proportional to the time evolution of $P_e(t)$ which was calculated in the previous subchapters:
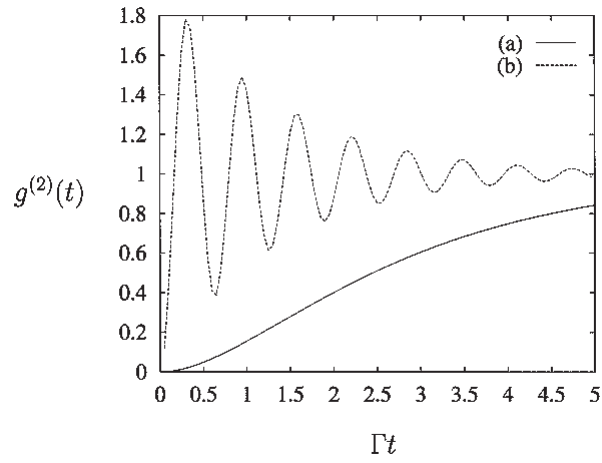
Figure 60: g$^{(2)}$-function for $4\Omega/\Gamma = 0, 1$ (solid) and $4\Omega/\Gamma = 10$ (dashed) [from Scully "Quantum Optics"]

$$P_e(t) = \frac{1}{2}\left(1 - e^{-3\gamma t/4}\cos\Omega t\right) \tag{454}$$

First experiments of the photon statistics from the fluorescence light of a single atom have been studied by the groups of Kimble (Kimble et al., Phys. Rev. Lett. 39, 691 (1977)) and Walther (Diedrich and Walther, Phys. Rev. Lett. 58, 203 (1987).
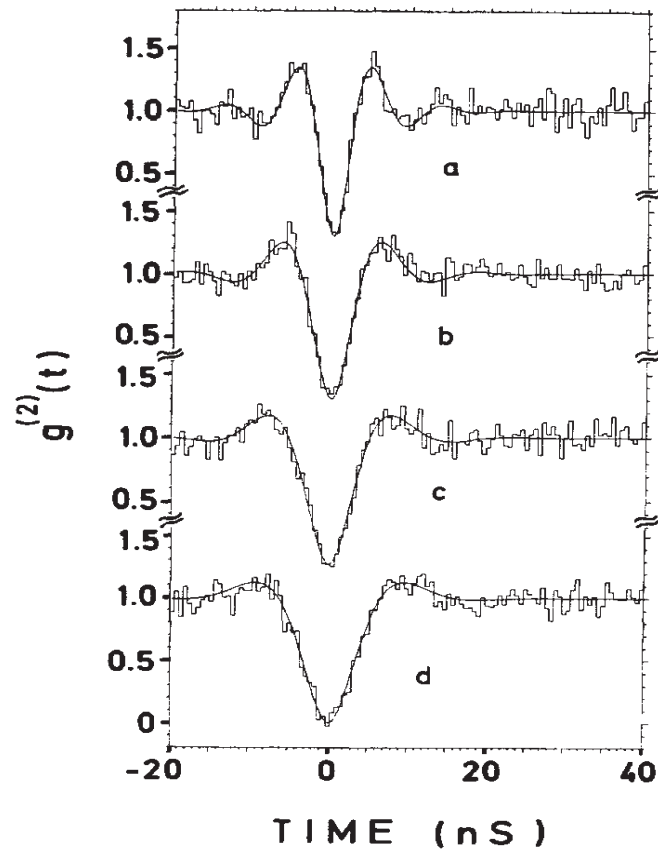
Figure 61: Measured intensity correlation of the fluorescence light from a single atom [Diedrich and Walther 1987]