QUANTUM OPTICS
Sommersemester 2008

Blatt 1
zur Übung am 24. April 2008

1. Commutator relations

Based on the quantum mechanical operators of the quantum mechanical harmonic oscillator

$$\hat{q} = \left(\frac{\hbar}{2m\omega}\right)^{1/2} (\hat{a} + \hat{a}^\dagger) \ ,$$

$$\hat{p} = -i \left(\frac{\hbar m\omega}{2}\right)^{1/2} (\hat{a} - \hat{a}^\dagger)$$

as well as their commutator

$$[\hat{q}, \hat{p}] = i\hbar$$

show that

$$[\hat{a}, \hat{a}^\dagger] = 1 \ .$$

Show also the following relations:

$$[\hat{a}, (\hat{a}^\dagger)^n] = n(\hat{a}^\dagger)^{n-1} \ ,$$

$$[\hat{a}^n, \hat{a}^\dagger] = n\hat{a}^{n-1} \ .$$

2. Quantization of the electromagnetic field

It is well known that for a classical electromagnetic field the energy is given by

$$H = \frac{1}{2} \int (\epsilon_0 \mathbf{E}^2 + \mu_0 \mathbf{H}^2) d^3 r \ ,$$

with

$$\begin{aligned}
\hat{\mathbf{E}} &= \sum_{\mathbf{k}}\sum_{\lambda}\left(\frac{\hbar\omega_{\mathbf{k}}}{2\epsilon_0 V}\right)^{1/2}\mathbf{e}_{\mathbf{k}\lambda}(\hat{a}_{\mathbf{k}\lambda}e^{-i(\omega_{\mathbf{k}}t-\mathbf{kr})}+c.c.)\ , \\
\hat{\mathbf{H}} &= \frac{1}{\mu_0}\sum_{\mathbf{k}}\sum_{\lambda}\left(\frac{\hbar}{2\omega_{\mathbf{k}}\epsilon_0 V}\right)^{1/2}(\mathbf{k}\times\mathbf{e}_{\mathbf{k}\lambda})(\hat{a}_{\mathbf{k}\lambda}e^{-i(\omega_{\mathbf{k}}t-\mathbf{kr})}+c.c.)
\end{aligned}$$

as well as

$$[\hat{a}_m, \hat{a}_n^\dagger] = \delta_{m,n}$$

show that for a quantum mechanical field the energy is given by

$$\hat{H} = \sum_{\mathbf{k}}\sum_{\lambda}\hbar\omega_{\mathbf{k}}(\hat{a}_{\mathbf{k}\lambda}^\dagger\hat{a}_{\mathbf{k}\lambda}+\frac{1}{2})\ .$$

3. Variance of the electrical field

Show that the variance of the electrical field of a fock state $|n_k\rangle$ equals

$$\langle\hat{\mathbf{E}}^2 - \langle\hat{\mathbf{E}}\rangle^2\rangle = \frac{\hbar\omega_k}{\epsilon_0 V}(n_k+\frac{1}{2})\ .$$