QUANTUM OPTICS
Sommersemester 2008

Blatt 4
Abgabe am 20. Mai 2008

1. P-representation I

The P-representation is an expansion of the density matrix ρ in the coherent state basis $\{|\alpha\rangle\}$:

$$\rho = \int P(\alpha, \alpha^*)|\alpha\rangle\langle\alpha| d^2\alpha.$$

The 2D Fourier transform $\tilde{f}(\beta)$ of a function $f(\alpha)$ of complex variables $\alpha = x_\alpha + iy_\alpha$, $\beta = x_\beta + iy_\beta$ is given by

$$\begin{aligned} \tilde{f}(\beta) &= \int\int f(\alpha) e^{2i(x_\alpha y_\beta - x_\beta y_\alpha)} dx_\alpha dy_\alpha, \\ f(\alpha) &= \frac{1}{\pi^2} \int\int \tilde{f}(\beta) e^{2i(x_\beta y_\alpha - x_\alpha y_\beta)} dx_\beta dy_\beta. \end{aligned}$$

Show that $P(\alpha, \alpha^*)$ can be derived as the Fourier transform of the matrix element $\langle -\beta|\rho|\beta\rangle$, i.e.

$$\begin{aligned} P(\alpha, \alpha^*) &= \frac{e^{x_\alpha^2 + y_\alpha^2}}{\pi^2} \int\int \langle -\beta|\rho|\beta\rangle e^{x_\beta^2 + y_\beta^2} e^{2i(y_\alpha x_\beta - x_\alpha y_\beta)} dx_\beta dy_\beta \\ &= \frac{e^{|\alpha|^2}}{\pi^2} \int \langle -\beta|\rho|\beta\rangle e^{|\beta|^2} e^{-\beta\alpha^* + \beta^*\alpha} d^2\beta. \end{aligned} \tag{1}$$

(Hint: use the relation $\langle\alpha|\alpha'\rangle = e^{-\frac{1}{2}|\alpha|^2 + \alpha^*\alpha' - \frac{1}{2}|\alpha'|^2}$.)

2. P-representation II

With relation (1) find $P(\alpha, \alpha^*)$ of

a) a coherent state $|\alpha\rangle$.

b) a Fock state $|n\rangle$.

3. Wigner function

The Wigner function $W(q,p)$ is the quasi-probability distribution of the conjugate variables q and p. The Wigner functions for coherent states and Fock states are given by

$$\begin{aligned} W_{coh}(Q,P) &= \frac{2}{\pi} e^{-\frac{1}{2}(Q^2+P^2)} \text{ and} \\ W_{Fock}(Q,P) &= \frac{2}{\pi}(-1)^n L_n(4(Q^2+P^2)) e^{-2(Q^2+P^2)}, \end{aligned}$$

respectivly, with $Q \propto q$, $P \propto p$ and $L_n(x)$ beeing the Laguerre polynomial.

Plot $W(Q,P)$ for a coherent state and a single photon Fock state. Where do these functions show non-classical behaviour?