8 Quantized Interaction of Light and Matter

8.1 Dressed States

Before we start with a fully quantized description of matter and light we would like to discuss the evolution of a two-level atom interacting with a *classical* field $E(t) = E_0 \exp(i\omega_1 t)$ in a different way.

As discussed in the previous section, the most general state of a two-level atom is:

$$|\psi\rangle = c_g e^{\frac{i}{2}\omega_1 t}|g\rangle + c_n e^{-\frac{i}{2}\omega_1 t}|e\rangle \tag{332}$$
$$= \tilde{c}_g|g\rangle + \tilde{c}_n|e\rangle \tag{333}$$
$$\tag{334}$$

where we have already transformed the coefficients into a rotating frame.
If we plug this state in the Schroedinger equation

$$|\dot{\psi}\rangle = \frac{1}{i\hbar}\left(H_A + H_I\right)|\psi\rangle \tag{335}$$

with $H_A = \frac{1}{2}\hbar\omega_0\sigma_z$ and $H_I = -\vec{\mu}\vec{E}$
we find coupled equations for the coefficients $\tilde{c}_e(t)$ and $\tilde{c}_g(t)$.

$$\dot{\tilde{c}_e} = i\left(-\frac{\delta}{2}\tilde{c}_e + \hbar^{-1}\langle e|H_I|g\rangle\tilde{c}_g\right) \tag{336}$$

$$\dot{\tilde{c}_g} = i\left(\frac{\delta}{2}\tilde{c}_g + \hbar^{-1}\langle g|H_I|e\rangle\tilde{c}_e\right) \tag{337}$$

We can write this equation in matrix form:

$$\begin{pmatrix} \dot{\tilde{c}_e} \\ \dot{\tilde{c}_g} \end{pmatrix} = \begin{pmatrix} -\frac{\delta}{2} & \Omega/2 \\ \Omega/2 & \frac{\delta}{2} \end{pmatrix} \begin{pmatrix} \tilde{c}_e \\ \tilde{c}_g \end{pmatrix} \tag{338}$$

with the Rabi-frequency $\Omega = 2|\hbar^{-1}\langle e|H_I|g\rangle|$

The solution of these equations are again the Bloch-equations (without damping!) as derived in the previous chapter. We now start with a diagonalization of the 2x2 Hamiltonian.

The eigenenergies of the Hamiltonian are:

$$E_{1,2} = \pm\frac{1}{2}\sqrt{\delta^2 + \Omega^2} = \pm\frac{1}{2}\tilde{\Omega} \tag{339}$$

The two eigenenergies differ by the generalized Rabi-frequency.

The according eigenstates are:

$$|1\rangle = \sin\theta|e\rangle + \cos\theta|g\rangle \tag{340}$$

$$|2\rangle = \cos\theta|e\rangle - \sin\theta|g\rangle \tag{341}$$

where

$$\tan\theta = \frac{\Omega}{\tilde{\Omega} - \delta} \tag{342}$$

These states are called **dressed states**. They are superpositions of the uncoupled **bare states**.

The following figure shows the energy structure of the eigenstates for different detunings in the case without coupling and with coupling.

In exact resonance ($\delta = 0$) the states would be degenerate in the case without coupling. However, a classical field couples the two bare states which mix and form the dressed states. Instead of a crossing a pronounced anti-crossing is observed. The splitting is proportional to the Rabi-frequency and hence to the strength of the classical field. Far from resonance both eigenstates approximate the bare states. The shift of the eigenenergies at the anti-crossing is also denoted as *AC stark shift*.

As explained, the former bare states are no eigenstates of the coupled Hamiltonian. Their dynamical behavior can now easily be derived from the new eigenstates, the dressed states:
Obviously, the bare states can be written as coherent superpositions in the dressed

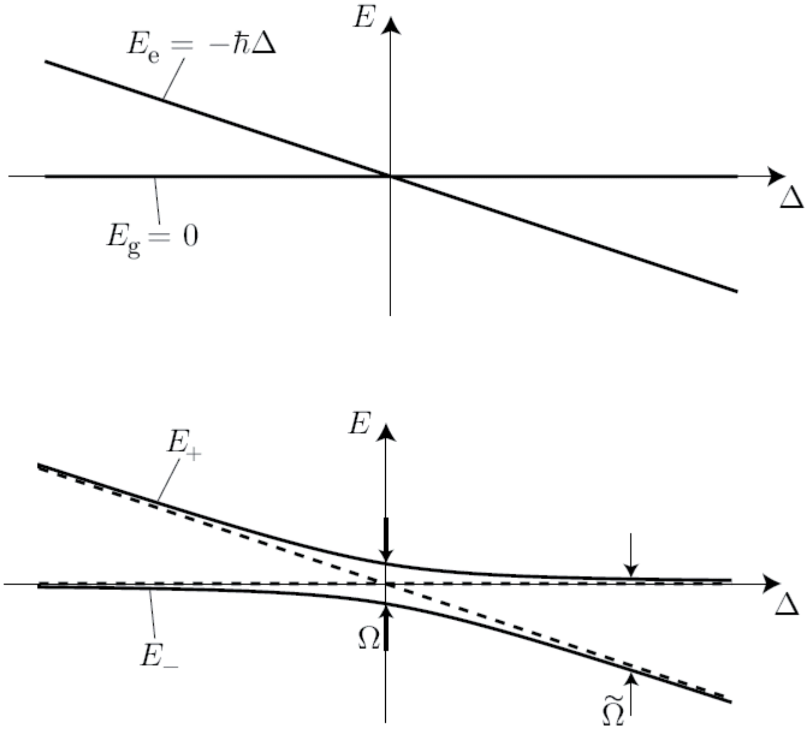

Figure 54: Eigenenergies as a function of the detuning Δ. Top: in case of zero coupling (no classical field present); Bottom: in case of coupling via a classical field. [from D.A. Steck *Quantum and Atom Optics*]. Note: In this figure one eigenenergy was kept at a fixed reference level.

states basis:

$$|e\rangle = |1\rangle + |2\rangle \tag{343}$$
$$|g\rangle = |1\rangle - |2\rangle \tag{344}$$

The phase of the basis states now starts to evolve, e.g. for the state $|e\rangle$ and $\delta = 0$:

$$|\psi\rangle = e^{-iE_1 t/\hbar}|1\rangle + e^{-iE_2 t/\hbar}|2\rangle \tag{345}$$
$$= e^{-i\Omega t/2}|1\rangle + e^{i\Omega t/2}|2\rangle \tag{346}$$
$$= e^{-i\Omega t}\left(|1\rangle + e^{i\Omega t}|2\rangle\right) \tag{347}$$

Apart from an overall phase, at $\Omega t = 2n\pi$ the state $|\psi\rangle = |e\rangle$, but at $\Omega t = (2n+1)\pi$ the state is $|\psi\rangle = |g\rangle$. This dynamical behavior is the known Rabi flopping.

8.2 Interaction of an atom with a quantized field

Now we will describe the interaction of an atom with a quantum field. It is convenient to start from the Hamiltonian:

$$H = \frac{1}{2m}(p - eA)^2 + eV(x) + H_{field} \tag{348}$$
$$= H_A + H_I + H_{field} \tag{349}$$

where

$$H_A = \int \psi^+(x) \left(-\frac{\hbar^2}{2m}\nabla^2 + eV(x) \right) \psi(x)\ dx \tag{350}$$
$$H_I = \int \psi^+(x) \left(-\frac{e}{m}Ap + \frac{e^2}{2m}A^2 \right) \psi(x)\ dx \tag{351}$$

The last term in H_I can usually be neglected for not too intense fields.

Inserting the expression for the quantized vector potential gives:

$$H_A = \sum_j E_j b_j^+ b_j \tag{352}$$
$$H_I = \hbar \sum_{j,k,\lambda} b_j^+ b_k \left(g_{\lambda jk} a_\lambda + g^*_{\lambda jk} a^+_\lambda \right) \tag{353}$$

with

$$g_{\lambda jk} = i\frac{e}{m}\sqrt{\frac{1}{2\hbar\omega_\lambda \epsilon_0}} \int \psi_j^*(x)\left(u_\lambda(x)p\right)\psi_k(x)\ dx \tag{354}$$

If $u_\lambda(x)$ varies much more slowly than the extension of the electronic wavefunction ($\lambda_{photon} \gg r_{atom}$, typically $\lambda/r \approx 10^3$) then $u_\lambda(x)$ can be taken out of the integral. In this *electric dipole approximation* one finds

$$\int \psi_j^*(x) p \psi_k(x)\ dx = \frac{im}{\hbar}\int \psi_j^*(x)\left[H_A, x\right]\psi_k(x)\ dx \tag{355}$$
$$= \frac{im}{\hbar}\left(E_j - E_k\right)\int \psi_j^*(x) x \psi_k(x)\ dx \tag{356}$$
$$= im\omega_0 m_{12} \tag{357}$$

Therefore, one can write:

$$H = H_A + H_I + H_{field} \quad \text{with} \tag{358}$$
$$H_A = \sum_j E_j b_j^+ b_j \tag{359}$$
$$H_{field} = \sum_k \hbar\omega_k a_k^+ a_k \tag{360}$$
$$H_I = \hbar \sum_{j,k,\lambda} g_{jk\lambda} b_j^+ b_k (a_\lambda + a_\lambda^+) \tag{361}$$

From the solution of the unperturbed Hamiltonian it can be seen that $b_j^+, b_k, a_\lambda, a_\lambda^+$ oscillate rapidly with optical frequencies (e.g. $b(t) = 1/i\hbar\,[H, b]$):

$$b_k = b_k(0)e^{-iE_k t/\hbar} \tag{362}$$
$$b_j^+ = b_j^+(0)e^{iE_j t/\hbar} \tag{363}$$
$$a_\lambda = a_\lambda(0)e^{-i\omega_\lambda t} \tag{364}$$

For not too intense fields only resonant terms with $\omega_0 = (E_j - E_i)/\hbar \simeq \omega_\lambda$ and the form $\exp i(\omega_{ij} - \omega_\lambda)t$ are significant in the dynamics.

In this *rotating wave approximation* the Hamiltonian for the *two-level atom* interacting with a quantized field is:

$$H = H_0 + H_I \tag{365}$$
$$H_0 = \frac{1}{2}\hbar\omega_0\sigma_z + \sum_k \hbar\omega_k a_k^+ a_k \tag{366}$$
$$H_I = \hbar\sum_\lambda g_\lambda (a_\lambda\sigma^+ + a_\lambda^+\sigma^-) \tag{367}$$

with

$$g_\lambda = -\left(\frac{1}{2\hbar\varepsilon_0\omega_\lambda}\right)^{1/2} \omega_0 u_\lambda(x_0)\mu_{12} \tag{368}$$

with the *dipole moment* $\mu_{12} = em_{12}$

Or if $\omega_\lambda \approx \omega_0$ then

$$g_\lambda = \sqrt{\frac{\omega_0}{2\hbar\varepsilon_0}} u_\lambda(x_0)\mu_{12} \tag{369}$$

$$= \Omega_0/2 \tag{370}$$

with the *vacuum Rabi frequency*

$$\Omega_0 = 2\frac{1}{\hbar}\sqrt{\frac{\hbar\omega_0}{2\varepsilon_0}} u_\lambda(x_0)\mu_{12} = 2\frac{1}{\hbar}\sqrt{\frac{\hbar\omega_0}{2\varepsilon_0 V}} \tilde{u}_\lambda(x_0)\mu_{12} = \frac{2E_0\mu_{12}}{\hbar}\tilde{u}_\lambda(x_0) \tag{371}$$

which is similar as in the classical case, but with the classical field replaced by the electric field per photon and explicitly taking into account the mode function $\tilde{u}_\lambda(x_0)$.

8.3 Jaynes-Cummings Model

The most simple case occurs if a single two-level atom interacts with a single mode of the electromagnetic field.

For this case the *Jaynes-Cummings-Hamiltonian* applies:

$$H_{JC} = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega a^+ a + \hbar g(a\sigma^+ + a^+\sigma^-) \tag{372}$$

This Hamiltonian only couples states $|n, e\rangle$ with $|n+1, g\rangle$ where we denote with $|e\rangle, |g\rangle$ the excited and ground state of the two-level atom.

It thus suffices to describe H in this basis and define:

$$H_n = \hbar\left(n + \frac{1}{2}\right)\omega \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \hbar \begin{pmatrix} \delta/2 & g\sqrt{n+1} \\ g\sqrt{n+1} & -\delta/2 \end{pmatrix} \tag{373}$$

with $\delta = \omega_0 - \omega$.

The eigenenergies of this Hamiltonian are:

$$E_{2n} = \hbar\left(n + \frac{1}{2}\right)\omega - \frac{1}{2}\hbar\Omega_n \tag{374}$$

$$E_{1n} = \hbar\left(n + \frac{1}{2}\right)\omega + \frac{1}{2}\hbar\Omega_n \tag{375}$$

with

$$\Omega_n = \sqrt{\delta^2 + \Omega_0^2 (n+1)} \tag{376}$$

Ω_n is called the generalized Rabi frequency.

The according eigenstates are:

$$|2n\rangle = \cos\vartheta_n |e,n\rangle - \sin\vartheta_n |g,n+1\rangle \tag{377}$$
$$|1n\rangle = \sin\vartheta_n |e,n\rangle + \cos\vartheta_n |g,n+1\rangle \tag{378}$$

with

$$\cos\vartheta_n = \frac{\Omega_n - \delta}{\sqrt{(\Omega_n - \delta)^2 + 4g^2 (n+1)}} \tag{379}$$
$$\sin\vartheta_n = \frac{2g\sqrt{(n+1)}}{\sqrt{(\Omega_n - \delta)^2 + 4g^2 (n+1)}} \tag{380}$$

These eigenstates of the combined atom-field system are called *dressed states*. The quantized expressions are identical to the semiclassical expression with $\Omega_0\sqrt{n+1}$ replaced by its semiclassical expression Ω. A striking difference is that even the vacuum filed ($n = 0$, i.e. zero field amplitude) can couple the two states. this is why Ω_0 is called *vacuum Rabi-frequency.*

On resonance the dressed states reduce to:

$$|2n\rangle = (|e,n\rangle - |g,n+1\rangle)/\sqrt{2} \tag{381}$$
$$|1n\rangle = (|e,n\rangle + |g,n+1\rangle)/\sqrt{2} \tag{382}$$

with eigenenergies:

$$E_{2n} = \hbar\left(n + \frac{1}{2}\right)\omega - \hbar g\sqrt{n+1} \tag{383}$$
$$E_{1n} = \hbar\left(n + \frac{1}{2}\right)\omega + \hbar g\sqrt{n+1} \tag{384}$$

It is easy to show that in the interaction picture (rotating at the frequency $(n+1/2)\omega$)

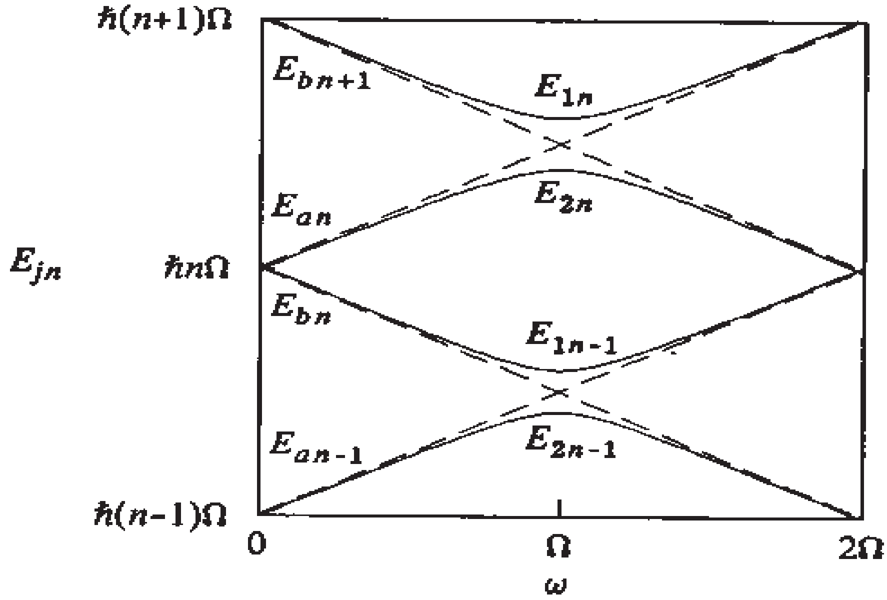

Figure 55: Dressed states. Dashed lines show energy levels without coupling. [from Meystre "Elements of Quantum Optics"]

the coefficients $c_{1n}(t), c_{2n}(t)$ of an arbitrary state $|\psi(t)\rangle = c_{1n}(t)\,|1n\rangle + c_{2n}(t)\,|2n\rangle$ obey:

$$\begin{pmatrix} c_{2n}(t) \\ c_{1n}(t) \end{pmatrix} = \begin{pmatrix} \exp(i\Omega_n t) & 0 \\ 0 & \exp(-i\Omega_n t) \end{pmatrix} \begin{pmatrix} c_{2n}(0) \\ c_{1n}(0) \end{pmatrix} \tag{385}$$

In the resonant case $\delta = 0$ this gives for a state initially in the upper state:

$$|c_{en}(t)|^2 = \cos^2(g\sqrt{n+1}t) \tag{386}$$

$$|c_{gn+1}(t)|^2 = \sin^2(g\sqrt{n+1}t) \tag{387}$$

Even if $n = 0$ (no photon or interaction with the vacuum) there is:

$$|c_{e0}(t)|^2 = \cos^2(gt) = \frac{1}{2}\left(1 + \cos(\Omega_0 t)\right) \tag{388}$$

Thus there is a coherent exchange of one energy quantum between the atom and the field mode, the so-called *vacuum Rabi oscillation*, in striking difference to the irreversible exponential decay into free space of an excited atom. The periodic energy exchange has an analogy with two coupled pendula.

The coupled equation of motion for the states $|e,n\rangle$ and $|g,n+1\rangle$ are:

$$\dot{c}_{en} = -i\frac{\delta}{2}c_{en} - ig\sqrt{n+1}c_{gn+1} \tag{389}$$

$$\dot{c}_{gn+1} = i\frac{\delta}{2}c_{gn+1} - ig\sqrt{n+1}c_{en} \tag{390}$$

Experiments to show the vacuum Rabi oscillations have been performed recently.

M. Brune, et al., Phys. Rev. Lett. 76, 1800-1803 (1996); B. T. H. Varcoe, S. Brattke, M. Weidinger, H. Walther, Nature 403, 743 - 746 (2000)

8.4 Wigner-Weisskopf theory of spontaneous emission

The Hamiltonian for a single two-level atom coupled to a discrete number of modes of an e.magn. field is:

$$H = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\sum_k \omega_k a_k^+ a_k + \hbar\sum_k g_k(a_k\sigma^+ + a_k^+\sigma^-) \tag{391}$$

The most general state vector is:

$$|\psi(t)\rangle = c_{e0}(t)\,|e\{0\}\rangle + \sum_k c_{g\{1_k\}}e^{-i(\omega_k-\omega_0)t}\,|g\{1_k\}\rangle \tag{392}$$

Substituting into the Schroedinger equation gives:

$$\dot{c}_{e0} = -i\sum_k g_k c_{g\{1_k\}}e^{-i(\omega_k-\omega_0)t} \tag{393}$$

$$\dot{c}_{g\{1_k\}} = -ig_k c_{e0}e^{i(\omega_k-\omega_0)t} \tag{394}$$

Formally integrating and inserting results to:

$$\dot{c}_{e0} = -\sum_k g_k^2 \int\limits_0^t dt' \; e^{-i(\omega_k-\omega_0)(t-t')} c_{e0}(t') \tag{395}$$

We now move from a discrete set of modes to a continuum by replacing the sum over k with an integral:

$$\sum_k f(k) \longrightarrow \frac{V}{(2\pi)^3}\int d^3k \; f(k) = \frac{V}{(2\pi c)^3}\int d\omega \; \omega^2 \int_0^\pi d\vartheta \sin\vartheta \int_0^{2\pi} d\varphi \; f(\omega,\vartheta,\varphi) \tag{396}$$

We also insert

$$g_k^2(\omega,\vartheta) = \frac{1}{\hbar^2}\sum_{\sigma=1}^{2} \left|\langle e|\, e\overrightarrow{r}\widehat{\epsilon}_\sigma \,|g\rangle \, E_{0,\omega} u_\omega\right|^2 \tag{397}$$

$$= \frac{1}{\hbar^2} E_{0,\omega}^2 \mu_{12}^2 \sin^2\vartheta \left|\cos^2\varphi + \sin^2\varphi\right| \tag{398}$$

$$= \frac{1}{\hbar^2} E_{0,\omega}^2 \mu_{12}^2 \sin^2\vartheta \tag{399}$$

$$= \frac{1}{\hbar^2}\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)\mu_{12}^2 \sin^2\vartheta \tag{400}$$

Inserting and integrating gives:

$$\dot{c}_{e0} = -\frac{1}{6\varepsilon_0 \pi^2 \hbar c^3}\int d\omega \; \omega^3 \mu_{12} \int\limits_0^t dt' \; e^{-i(\omega_k-\omega_0)(t-t')} c_{e0}(t') \tag{401}$$

with

$$\lim_{t\to\infty}\int\limits_0^t dt' \; e^{-i(\omega_k-\omega_0)(t-t')} = \pi\delta(\omega-\omega_0) - P\left[\frac{i}{\omega-\omega_0}\right] \tag{402}$$

it follows:

$$\dot{c}_{e0} = -\frac{\Gamma}{2} c_{e0}(t) \tag{403}$$

where the Lamb-shift is neglected.

The rate Γ is the Wigner-Weisskopf rate of spontaneous emission:

$$\Gamma = \frac{\omega^3 \mu_{12}^2}{3\pi\varepsilon_0 \hbar c^3} \tag{404}$$

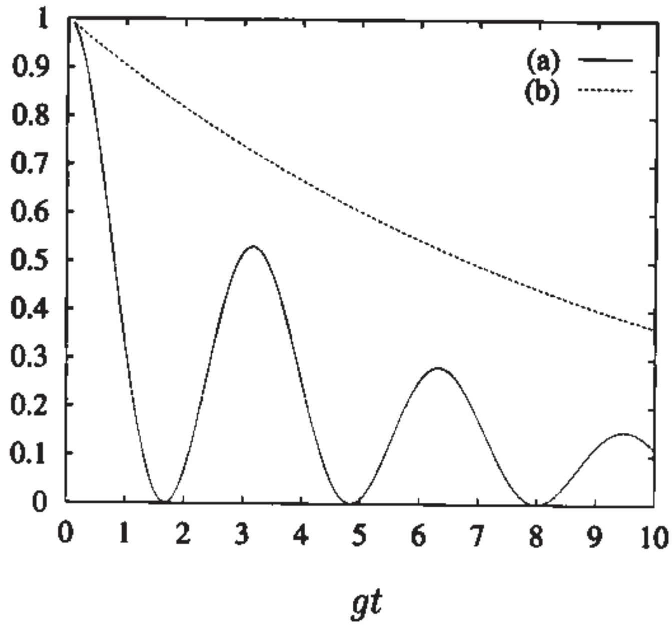

Figure 56: Probability to find an excited atom in a cavity in the upper state for weak damping (a) and strong damping (b) of the cavity field. [from Scully "Quantum optics"]

8.5 Collapse and Revival & Quantum beats

8.5.1 Collapse & Revival

An interesting phenomenon exists if a single atom interacts not with a single Fock-state $|n\rangle$, but with a coherent state $|\alpha\rangle$ where

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{405}$$

In this case the probability to find the initially excited atom in the excited state after some time t is:

$$P_e = \sum_n p_n \left|c_{en}(t)\right|^2 \tag{406}$$

$$= e^{-|\alpha|^2} \sum_n \frac{|\alpha|^{2n}}{n!} \cos^2(g\sqrt{n+1}t) \tag{407}$$

The time evolution is a sum of oscillations with different Rabi frequencies which then dephase.
This occurs on a timescale of appr.:

$$t_c \approx g^{-1} \tag{408}$$

However, after some time there is a *revival* of the probability to find the atom excited again. This is a pure quantum effect and due to the discrete number of basis states of the coherent state.

The time for the revival can be estimated to:

$$t_r \approx 4\pi\sqrt{\overline{n}} t_c = 4\pi\sqrt{\overline{n}} g^{-1} \tag{409}$$

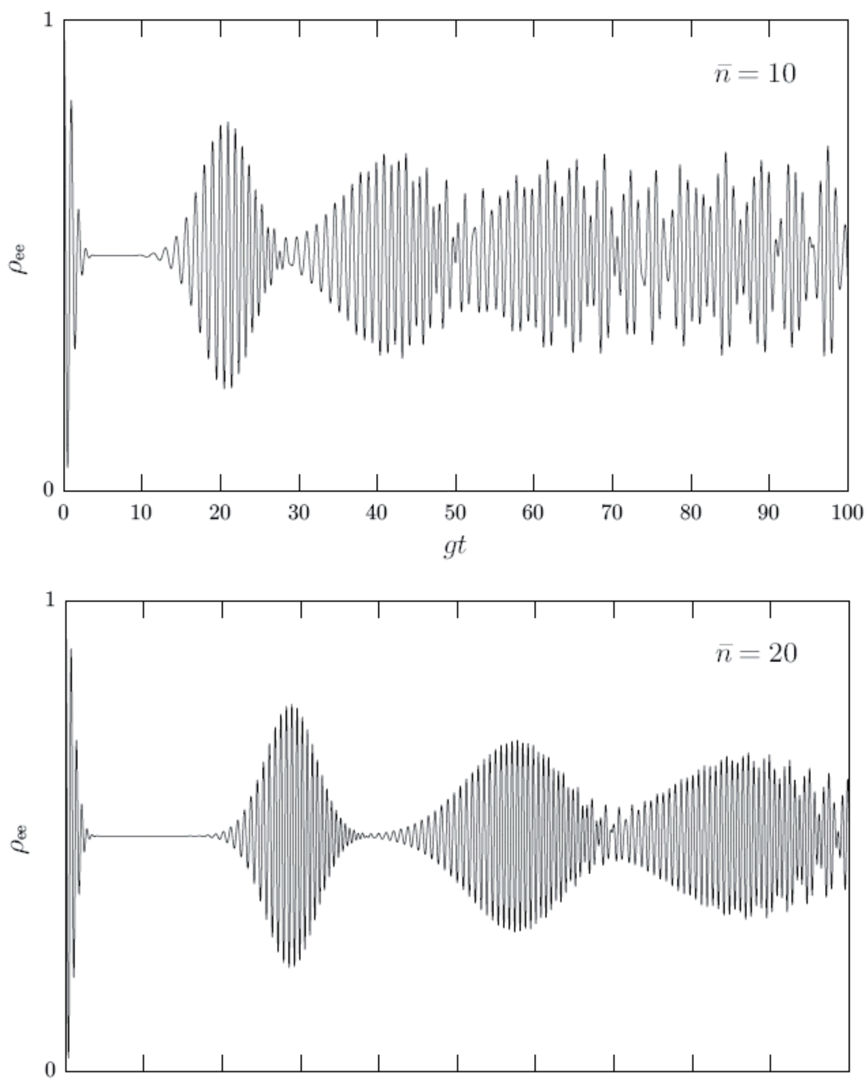

Figure 57: Collapse and revival for the interaction of a two-level system with a coherent state with $\overline{n} = 10$ (top) and $\overline{n} = 120$ [from D.A. Steck *Quantum and Atom Optics*]

8.5.2 Quantum Beats

Another interesting quantum effect in the spontaneous emission of light from a single atom is the quantum beat effect.

Consider the following Λ- and V-type three level systems:

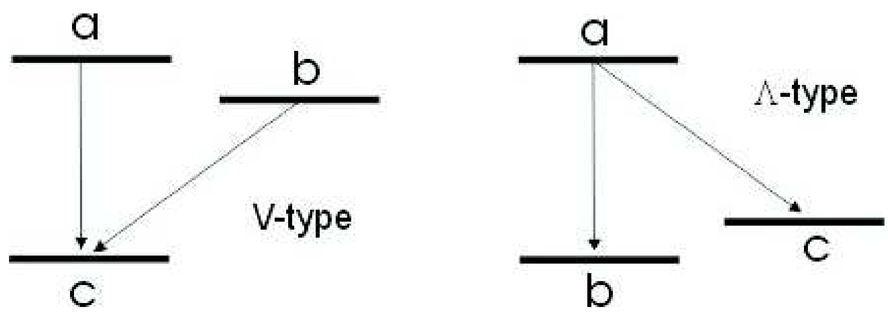

Figure 58: V-type and Λ-type three level systems

Assume the atomic state is in a superposition:

$$|\psi(t)\rangle = c_a e^{-i\omega_a t} |a\rangle + c_b e^{-i\omega_b t} |b\rangle + c_c e^{-i\omega_c t} |c\rangle \tag{410}$$

Semiclassically there exist oscillating dipoles:

V-type system:	P_{ac}	and	P_{bc}
Λ-type system:	P_{ab}	and	P_{ac}

which create a field of the form

$$E(t) = E_{01} \exp(-i\nu_1 t) + E_{02} \exp(-i\nu_2 t) \tag{411}$$

with

V-type system:	$\nu_1 = \omega_a - \omega_c$
	$\nu_2 = \omega_b - \omega_c$
Λ-type system:	$\nu_1 = \omega_a - \omega_b$
	$\nu_2 = \omega_a - \omega_c$

Obviously this creates a beating in a square law detector which can only measure intensities:

$$|E(t)|^2 = |E_{01}|^2 + |E_{02}|^2 + \{E_{01}^* E_{02} \exp\left[i\left(\nu_1 - \nu_2\right)t\right] + c.c.\} \tag{412}$$

for both the Λ- and the V-type system.

However, in the quantum case the beating signal is given by the following expressions:

V-type system:

$$I = \langle\psi_V(t)|\, E_1^{(-)} E_2^{(+)} \,|\psi_V(t)\rangle \tag{413}$$

with

$$E_1^{(-)} \propto a_1^+ e^{i\nu_1 t} \quad \text{and} \quad E_1^{(+)} \propto a_2 e^{-i\nu_2 t} \tag{414}$$

Therefore with the state

$$|\psi_V(t)\rangle = \sum_{i=a,b,c} c_i\,|i,0\rangle + c_1\,|c,1_{\nu 1}\rangle + c_2\,|c,1_{\nu 2}\rangle \tag{415}$$

this gives:

$$\begin{aligned} I &= const.\,\langle 1_{\nu 1} 0_{\nu 2}|\, a_1^+ a_2 \,|0_{\nu 1} 1_{\nu 2}\rangle \exp\left[i\left(\nu_1 - \nu_2\right)t\right] \langle c|c\rangle && (416)\\ &= const.\,\langle 1_{\nu 1} 0_{\nu 2}|\, a_1^+ a_2 \,|0_{\nu 1} 1_{\nu 2}\rangle \exp\left[i\left(\nu_1 - \nu_2\right)t\right] && (417)\end{aligned}$$

But, in the Λ-type system:

$$|\psi_\Lambda(t)\rangle = \sum_{i=a,b,c} c_i'\,|i,0\rangle + c_1'\,|b,1_{\nu 1}\rangle + c_2'\,|c,1_{\nu 2}\rangle \tag{418}$$

and

$$\begin{aligned} I &= \langle\psi_\Lambda(t)|\, E_1^{(-)} E_2^{(+)} \,|\psi_\Lambda(t)\rangle && (419)\\ &= const.\,\langle 1_{\nu 1} 0_{\nu 2}|\, a_1^+ a_2 \,|0_{\nu 1} 1_{\nu 2}\rangle \exp\left[i\left(\nu_1 - \nu_2\right)t\right] \langle c|b\rangle && (420)\\ &= 0 && (421)\end{aligned}$$

There is no beat note in the Λ-system.

This can be interpreted in the framework of which-path-information:

- After emission of a photon it is not possible to say which decay path the photon took in the V-system. Thus, the two paths interfere.
- In the Λ-system a measurement of the atomic state (it is either in $|b\rangle$ or $|c\rangle$) reveals information which path the photon took (even before it is detected). thus there is no interference.

This effect has some analogy to Young's double slit experiment.