Fundamentals of Optical Sciences

WS 2015/2016 **1 . Exercise** **19.10.2015**

Lecture: Prof. Dr. Alejandro Saenz, Prof. Dr. Oliver Benson

Prepare your answers for the excercise on 26.10.2015.

Exercise 1

The one-dimensional wave paket

$$p(z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}k\; A(k)\, e^{i(kz-\omega(k)t)} \tag{1}$$

in a dispersive medium with

$$\omega(k) = \alpha k^2 \tag{2}$$

possesses at $t = 0$ the shape of a Gaussian,

$$p(z,0) = C\, e^{-\frac{z^2}{2\Delta^2}}\, e^{ik_0 z}\,, \tag{3}$$

with the constants C, Δ, and k_0.

a) Show that the weight function

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}z\; p(z,0)\, e^{-ikz} \tag{4}$$

is also a Gaussian. What is the width Δk of $|A(k)|^2$? Let the width be defined as the distance Δk of the k values at which the function value has dropped to the fraction $1/e$ of its maximum.

Hint: $\displaystyle\int_{-\infty}^{+\infty} e^{-(ax^2+bx)}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}}$

b) Determine the full spatial and temporal dependence of $p(z,t)$.

c) Calculate the time dependence of the width $\Delta z(t)$ (defined analogous to Δk) of $|p(z,t)|^2$. What is the velocity with which the maximum of $|p(z,t)|^2$ moves? Compare it to the group velocity.

Exercise 2

A vector potential $\vec{A}(\vec{r},t)$ and a scalar potential $\phi(\vec{r},t)$ are given which should be transformed into a different gauge using the scalar field $\chi(\vec{r},t)$.

a) Which equation has to be satisfied by χ if the gauged potentials satisfy the Lorenz condition? Why is the Lorenz gauge denoted as a *gauge class* ?

b) Which equation has to be satisfied by χ to transform into the Coulomb gauge? Is this equation always solvable?

Exercise 3

Consider the following Maxwell equations with the electric field $\vec{E}$, the magnetic induction $\vec{B}$, the dielectric displacement $\vec{D}$, and the magnetic field strength $\vec{H}$,

$$\begin{aligned} \vec{\nabla}\cdot\vec{D} &= \rho_e & \vec{\nabla}\times\vec{H} &= \frac{\partial\vec{D}}{\partial t}+\vec{J}_e \\ \vec{\nabla}\cdot\vec{B} &= \rho_m & -\vec{\nabla}\times\vec{E} &= \frac{\partial\vec{B}}{\partial t}+\vec{J}_m\,, \end{aligned}$$

which allow for the existence of magnetic monopoles with the magnetic charge density $\rho_m(\vec{r})$ and the magnetic current density $\vec{J_m}(\vec{r})$.

a) Show that the Maxwell equations are invariant under the transformation

$$\begin{aligned} \vec{E} &= \vec{E}'\cos\xi + Z\vec{H}'\sin\xi & \vec{D} &= \vec{D}'\cos\xi + Z^{-1}\vec{B}'\sin\xi \\ \vec{H} &= -Z^{-1}\vec{E}'\sin\xi + \vec{H}'\cos\xi & \vec{B} &= -Z\vec{D}'\sin\xi + \vec{B}'\cos\xi \end{aligned}$$

(where $Z=\sqrt{\mu_0/\epsilon_0}$) under the condition that the charge density and the current density transform as

$$\begin{aligned} \rho_e &= \rho_e'\cos\xi + Z^{-1}\rho_m'\sin\xi & \vec{J}_e &= \vec{J}_e'\cos\xi + Z^{-1}\vec{J}_m'\sin\xi \\ \rho_m &= -Z\rho_e'\sin\xi + \rho_m'\cos\xi & \vec{J}_m &= -Z\vec{J}_e'\sin\xi + \vec{J}_m'\cos\xi\,. \end{aligned}$$

b) The Lorentz force acting on a particle with electric charge q_e and magnetic charge q_m is given by

$$\vec{F} = q_e\left(\vec{E}+\vec{v}\times\vec{B}\right) + q_m\left(\vec{H}-\vec{v}\times\vec{D}\right).$$

Show that the Lorentz force is invariant under the transformations above as well.

c) What are the consequences of the invariance of the transformations for the existence of magnetic monopoles?