Exercise 1

a) Show with the help of the (retarded) potentials from the first problem of the 3rd exercise sheet

\[ \phi(\vec{r}, t) = \frac{d_0 \cos \theta}{4\pi\varepsilon_0 r} \left( -\frac{\omega}{c} \sin \left[ \omega \left( t - \frac{r}{c} \right) \right] + \frac{1}{r} \cos \left[ \omega \left( t - \frac{r}{c} \right) \right] \right) \]

and

\[ \vec{A}(\vec{r}, t) = -\frac{\mu_0 \omega d_0}{4\pi r} \sin \left[ \omega \left( t - \frac{r}{c} \right) \right] \hat{e}_z, \]

and using \( r \gg c/\omega \), that the electric and the magnetic fields of a dipole oscillating along the \( z \) axis are

\[ \vec{E}(\vec{r}, t) = \frac{\mu_0\omega^2 d_0}{4\pi r} \cos \left[ \omega \left( t - \frac{r}{c} \right) \right] \left( \hat{e}_z - \frac{z}{r} \hat{e}_r \right) \]

and

\[ \vec{B}(\vec{r}, t) = \frac{1}{c} \left( \hat{e}_r \times \vec{E} \right). \]

Here \( \hat{e}_r = \vec{r}/r \) is the unit vector in radial direction.

**Hint:** Use the relations \( \hat{e}_z = \hat{e}_r \cos \theta - \hat{e}_\theta \sin \theta \) and \( \hat{e}_r \times \hat{e}_\theta = \hat{e}_\varphi \) to get the form \( \vec{E} \propto \hat{e}_\theta \) and \( \vec{B} \propto \hat{e}_\varphi \).

b) A rotating dipole \( \vec{d} \) can be represented as the superposition of two oscillating dipoles where one dipole oscillates along the \( x \) axis and the other one along the \( y \) axis. The phase difference between the two dipoles is \( \pi/2 \), i.e.

\[ \vec{d} = d_0 \left[ \cos(\omega t) \hat{e}_x + \sin(\omega t) \hat{e}_y \right]. \]

Determine the electric and the magnetic fields of an oscillating dipole using the superposition principle and the fields from part a). Calculate also the Poynting vector \( \vec{S} \), its time average \( \langle \vec{S} \rangle \), and the emitted power \( P \). Compare \( P \) with the emitted power

\[ P_1 = \frac{\mu_0 d_0^2 \omega^4}{12\pi c} \]

of an dipole oscillating along one axis. Did you expect your result?

**Hint:** \( \vec{a} \times \left( \vec{b} \times \vec{c} \right) = \vec{b} (\vec{a} \cdot \vec{c}) - \vec{c} (\vec{a} \cdot \vec{b}) \); \( \hat{e}_r = (x\hat{e}_x + y\hat{e}_y + z\hat{e}_z)/r \) \( \Rightarrow \vec{E} \cdot \hat{e}_r = ? \)
Exercise 2

Step-index and graded-index optical fibers

a) A step-index fiber has a radius of $a = 5\mu m$, core refractive index $n_1 = 1.45$ and a fractional refractive-index change $\Delta = 0.002$. Determine the shortest wavelength $\lambda_c$ for which the fiber is a single-mode waveguide. If the wavelength is changed to $\lambda_c/2$, identify the indices $(l,m)$ of all the guided modes.

b) Compare the numerical apertures of the step-index fiber from a) with a graded-index fiber with $n_1 = 1.45$, $\Delta = 0.002$, and a parabolic refractive-index profile ($p = 2$), so that $n^2(y) = n^2_0(1 - p^2y^2)$.

Exercise 3

Asymmetric planar waveguide

Examine the TE field in an asymmetric planar waveguide consisting of a dielectric slab of width $d$ and a refractive index $n_1$ placed on a substrate of lower refractive index $n_2$ and covered with a medium of refractive index $n_3$, where $n_3 < n_2 < n_1$.

a) Determine an expression for the maximum inclination angle $\theta$ of plane waves undergoing total internal reflection, and the corresponding numerical aperture (NA) of the waveguide.

b) Write an expression for the self-consistency condition.

c) Determine an approximate expression for the number of modes $M$ (valid when $M$ is very large).

d) For the parameters $n_1 = 1.35$, $n_2 = 1.32$, $n_3 = 1.3$, plot the values of the effective refractive index ($\beta/k_0$) as a function of the slab width normalized by the wavelength ($d/\lambda$) for the first 3 modes.