Fundamentals of Optical Sciences

WS 2015/2016 **8 . Exercise** **7.12.2015**

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Prepare your answers for the exercise on 14.12.2015.

Exercise 1

Consider phenomenological decay terms for the density matrix. If decays only happen into states different from the two levels g and e, they can be introduced phenomenologically in the wavefunction picture. Let ϕ_g and ϕ_e be the stationary field-free atomic wavefunctions with the eigenenergies $\epsilon_g = \hbar\omega_g$ and $\epsilon_e = \hbar\omega_e$. For exponentially decaying states with the rates γ_g and γ_e the coefficients c_g and c_e of the wavefunction

$$\Psi(\vec{r}, t) = c_g(t)\,\phi_g(\vec{r})\,\mathrm{e}^{-\mathrm{i}\omega_g t} + c_e(t)\,\phi_e(\vec{r})\,\mathrm{e}^{-\mathrm{i}\omega_e t}$$

after transforming to the rotating coordinate system become

$$\tilde{c}_g(t) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\delta t}\,c_g(t) \quad \text{and} \quad \tilde{c}_e(t) = \mathrm{e}^{+\frac{\mathrm{i}}{2}\delta t}\,c_e(t).$$

Here $\delta = \omega - \omega_{e,g}$ is the detuning with the laser frequency is ω and the atomic transition frequency $\omega_{e,g} = \omega_e - \omega_g$. Using the *rotating wave approximation* (RWA) gives

$$\frac{\mathrm{d}\tilde{c}_g(t)}{\mathrm{d}t} = -\frac{1}{2}\left[(\gamma_g + \mathrm{i}\,\delta)\,\tilde{c}_g(t) + \mathrm{i}\,\Omega_0\,\tilde{c}_e(t)\right]$$
$$\frac{\mathrm{d}\tilde{c}_e(t)}{\mathrm{d}t} = -\frac{1}{2}\left[\mathrm{i}\,\Omega_0\,\tilde{c}_g(t) + (\gamma_e - \mathrm{i}\,\delta)\,\tilde{c}_e(t)\right]$$

where $\Omega_0 = d_{e,g}E_0/\hbar$ is the Rabi frequency with the transition dipole moment $d_{e,g}$ and and electric field amplitude E_0. In the following, assume Ω_0 is real.

a) Show that the introduction of these decay terms in the amplitude equations valid in the absence of external fields (i.e. for $\delta = E_0 = 0$) yields the desired exponential decay of the coefficients.

b) i) Show that for $\gamma = \gamma_g = \gamma_e$ the perturbative solutions of the differential equations can alternatively be obtained by using the transformations $\tilde{c}'_g = \mathrm{e}^{\frac{1}{2}\gamma t}\tilde{c}_g$ and $\tilde{c}'_e = \mathrm{e}^{\frac{1}{2}\gamma t}\tilde{c}_e$ for the solutions without decays (for an arbitrary number of levels).

ii) In the two-level case the population probability of the excited state is

$$P_e(t) = |c_e(t)|^2 \approx \frac{\Omega_0^2}{4}\left(\frac{\sin\left[\frac{1}{2}(\omega_{e,g} - \omega)t\right]}{\frac{1}{2}(\omega_{e,g} - \omega)}\right)^2 \mathrm{e}^{-\gamma t}.$$

Discuss this solution compared to the one without decays.

c) Now assume that the excited state $|\, e\, \rangle$ decays with the rate γ to the state $|\, f\, \rangle$ emitting a photon (spontaneous emission). Initially (for $t = 0$), all atoms are in the ground state $|\, g\, \rangle$. The population of the excited state $|\, f\, \rangle$ is given by

$$P_f(t) = \gamma \int\limits_0^t |c_e(t')|^2 \, \mathrm{d}t'.$$

i) Motivate this equation.

ii) Calculate the spectral distribution function $P_f(t)$.

iii) Calculate the limit $P_f(t \to \infty)$.

Exercise 2

The eigenstates of the Pauli matrix $\boldsymbol{\sigma}_z$ form a complete orthonormal basis, $|\, + \,\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with eigenvalue $+1$ and $|\, - \,\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with eigenvalue -1. Using these eigenstates the Observables A, B, and C have the following matrix representation:

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}; \quad \mathbf{C} = \begin{pmatrix} 0 & 2\mathrm{i} \\ -2\mathrm{i} & 0 \end{pmatrix}.$$

The expectation values of the observables

$$\langle A \rangle = 2; \quad \langle B \rangle = \frac{1}{2}; \quad \langle C \rangle = 0$$

were measured for a certain spin state.

a) Determine the density matrix $\boldsymbol{\rho}$ of the spin state.

b) Is it a pure or a mixed spin state?

c) What is the probability of finding the spin eigenvalue $+\hbar/2$ after a measurement in z direction?
Reminder: $\mathbf{S}_z = \frac{\hbar}{2}\boldsymbol{\sigma}_z$

d) Calculate the expectation values of the Pauli matrices $\langle \sigma_x \rangle$, $\langle \sigma_y \rangle$, $\langle \sigma_z \rangle$.
Hint:

$$\boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \boldsymbol{\sigma}_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}; \quad \boldsymbol{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$