Exercise 1

Consider phenomenological decay terms for the density matrix. If decays only happen into states different from the two levels $g$ and $e$, they can be introduced phenomenologically in the wavefunction picture. Let $\phi_g$ and $\phi_e$ be the stationary field-free atomic wavefunctions with the eigenenergies $\epsilon_g = \hbar \omega_g$ and $\epsilon_e = \hbar \omega_e$. For exponentially decaying states with the rates $\gamma_g$ and $\gamma_e$, the coefficients $c_g$ and $c_e$ of the wavefunction

$$\Psi(\vec{r}, t) = c_g(t) \phi_g(\vec{r}) e^{-i \omega_g t} + c_e(t) \phi_e(\vec{r}) e^{-i \omega_e t}$$

after transforming to the rotating coordinate system become

$$\tilde{c}_g(t) = e^{-\frac{i}{2} \delta t} c_g(t) \quad \text{and} \quad \tilde{c}_e(t) = e^{\frac{i}{2} \delta t} c_e(t).$$

Here $\delta = \omega - \omega_{e,g}$ is the detuning with the laser frequency is $\omega$ and the atomic transition frequency $\omega_{e,g} = \omega_e - \omega_g$. Using the rotating wave approximation (RWA) gives

$$\frac{d \tilde{c}_g(t)}{dt} = -\frac{1}{2} \left[ (\gamma_g + i \delta) \tilde{c}_g(t) + i \Omega_0 \tilde{c}_e(t) \right]$$
$$\frac{d \tilde{c}_e(t)}{dt} = -\frac{1}{2} \left[ i \Omega_0 \tilde{c}_g(t) + (\gamma_e - i \delta) \tilde{c}_e(t) \right]$$

where $\Omega_0 = d_{e,g} E_0 / \hbar$ is the Rabi frequency with the transition dipole moment $d_{e,g}$ and and electric field amplitude $E_0$. In the following, assume $\Omega_0$ is real.

a) Show that the introduction of these decay terms in the amplitude equations valid in the absence of external fields (i.e. for $\delta = E_0 = 0$) yields the desired exponential decay of the coefficients.

b) i) Show that for $\gamma = \gamma_g = \gamma_e$ the perturbative solutions of the differential equations can alternatively be obtained by using the transformations $\tilde{c}'_g(t) = e^{\frac{i}{2} \gamma t} \tilde{c}_g(t)$ and $\tilde{c}'_e(t) = e^{\frac{i}{2} \gamma t} \tilde{c}_e(t)$ for the solutions without decays (for an arbitrary number of levels).

ii) In the two-level case the population probability of the excited state is

$$P_e(t) = |c_e(t)|^2 \approx \frac{\Omega_0^2}{4} \left( \sin \left[ \frac{1}{2} (\omega_{e,g} - \omega) t \right] \right)^2 e^{-\gamma t}.$$ 

Discuss this solution compared to the one without decays.
c) Now assume that the excited state $|e\rangle$ decays with the rate $\gamma$ to the state $|f\rangle$ emitting a photon (spontaneous emission). Initially (for $t = 0$), all atoms are in the ground state $|g\rangle$. The population of the excited state $|f\rangle$ is given by

$$P_f(t) = \gamma \int_0^t |c_e(t')|^2 \, dt'.$$

i) Motivate this equation.

ii) Calculate the spectral distribution function $P_f(t)$.

iii) Calculate the limit $P_f(t \to \infty)$.

Exercise 2

The eigenstates of the Pauli matrix $\sigma_z$ form a complete orthonormal basis, $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with eigenvalue $+1$ and $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with eigenvalue $-1$. Using these eigenstates the Observables $A$, $B$, and $C$ have the following matrix representation:

$$A = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}; \quad C = \begin{pmatrix} 0 & 2i \\ -2i & 0 \end{pmatrix}.$$  

The expectation values of the observables

$$\langle A \rangle = 2; \quad \langle B \rangle = \frac{1}{2}; \quad \langle C \rangle = 0$$

were measured for a certain spin state.

a) Determine the density matrix $\rho$ of the spin state.

b) Is it a pure or a mixed spin state?

c) What is the probability of finding the spin eigenvalue $+\hbar/2$ after a measurement in $z$ direction?

*Reminder:* $S_z = \frac{\hbar}{2} \sigma_z$

d) Calculate the expectation values of the Pauli matrices $\langle \sigma_x \rangle$, $\langle \sigma_y \rangle$, $\langle \sigma_z \rangle$.

*Hint:*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$  

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