Fundamentals of Optical Sciences

WS 2015/2016 **10 . Exercise** **04.01.2016**

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Prepare your answers for the exercise on 11.01.2016.

Exercise 1

Consider the eigenstates $|\,n\,\rangle$ of the one-dimensional harmonic oscillator (with mass m and frequency ω) and the associated creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$.

a) Proof with the aid of the creation and annihilation operators the following recursion relations

$$\begin{aligned}\hat{q}\,|\,n\,\rangle &= \sqrt{\frac{\hbar}{2\,m\,\omega}}\,\left[\sqrt{n+1}\,|\,n+1\,\rangle + \sqrt{n}\,|\,n-1\,\rangle\right]\\ \hat{p}\,|\,n\,\rangle &= i\sqrt{\frac{m\,\hbar\,\omega}{2}}\,\left[\sqrt{n+1}\,|\,n+1\,\rangle - \sqrt{n}\,|\,n-1\,\rangle\right]\end{aligned}$$

with the position operator $\hat{q} = \sqrt{\hbar/(2m\omega)}\,(\,\hat{a} + \hat{a}^\dagger\,)$ and the momentum operator $\hat{p} = -i\,\sqrt{m\hbar\omega/2}\,(\,\hat{a} - \hat{a}^\dagger\,)$.

b) Calculate (with the aid of the creation and annihilation operators) the expectation values of

i) the position operator $\hat{q}$,

ii) the momentum operator $\hat{p}$,

iii) the square of the position operator $\hat{q}^2$ and

iv) the square of the momentum operator $\hat{p}^2$

for the states $|\,n\,\rangle$.

c) Is the relation

$$\langle\,n\,|\,(\hat{a}^\dagger)^3\,\hat{a}^4\,\hat{a}^\dagger\,|\,n\,\rangle = \frac{(n-1)\,(n-2)}{(n+2)\,(n+3)}\,\langle\,n\,|\,\hat{a}^3\,(\hat{a}^\dagger)^4\,\hat{a}\,|\,n\,\rangle$$

valid?

d) Calculate the commutator relations

i) $[\hat{n}, \hat{a}]$,

ii) $[\hat{n}, \hat{a}^\dagger]$,

iii) $[\hat{a}, (\hat{a}^\dagger)^n]$,

iv) $[(\hat{a})^n, \hat{a}^\dagger]$ and

v) $[\hat{a}, \exp(\beta\, \hat{a}^\dagger)]$

with the number operator $\hat{n} = \hat{a}^\dagger\, \hat{a}$ and a constant β.

Exercise 2

Consider a two-mirror cavity with a gain medium. The mirrors have intensity reflection coefficients R_1 and R_2. The gain medium is thin and has a round-trip intensity gain coefficient G.

a) Write down the cavity intensity at time $t + \tau_{\mathrm{rt}}$ in terms of the cavity intensity at time t, where τ_{rt} is the round-trip time of the cavity.

b) Expand the above expression to first order in τ_{rt}, and hence obtain a differential equation for $I(t)$. What are the conditions under which this approximation is justified?

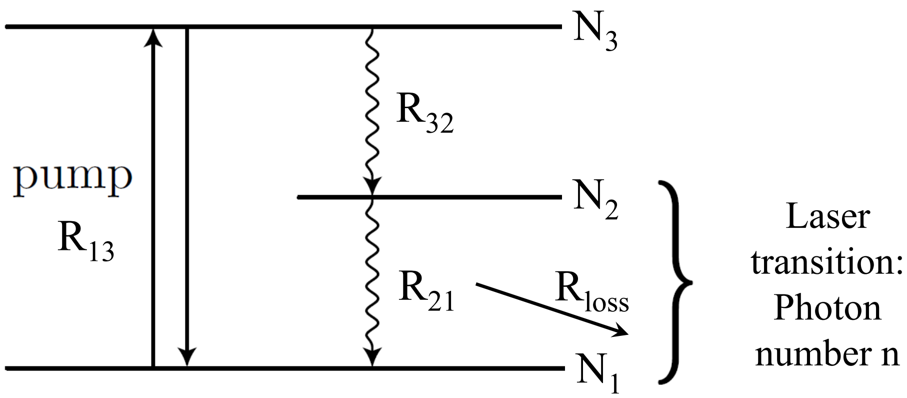
c) Apply the result of part b) to a cavity without a gain medium, to derive an expression for the cavity photon lifetime τ_{rt}, defined as the time required for the cavity intensity to decay to $1/e$ times its initial value.

Exercise 3

Write the rate equations for a 2-level system, showing that a steady-state population inversion cannot be achieved by using direct optical pumping between levels 1 and 2.

Exercise 4

Consider the three-level laser scheme shown in the figure. The laser-cavity interacts only with transitions between level 2 and 1 (with the population numbers N_2 and N_1). The cavity looses photons with a rate R_{loss}.



a) Write down the rate equations for the three levels of this gain-medium and the photon number n. Only the ratio of $0 < \beta < 1$ of R_{21} actually "feeds" the cavity (laser transition).

b) What are the conditions on R_{13}, R_{32}, R_{21} and β for a given R_{loss} that ensure the best laser operation?

c) What is the average photon number in the cavity that ensures that there are at least as many stimulated decay processes (from level 2 to 1) than spontaneous?

d) Take a look at the rate-equation for the photon number in the stationary case in the lasing regime (ignore spontaneous emission): Derive an equation for the inversion $D{=}N_2 - N_1$.