Fundamentals of Optical Sciences

WS 2015/2016 — 6 . Exercise — 23.11.2015

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Prepare your answers for the exercise on 30.11.2015.

Exercise 1

Consider a two-level atom and its interaction with the electric component of an electromagnetic field $\vec{E}(t) = E_0 \cos(\omega t)\,\vec{\varepsilon}$ in the semiclassical description. The full Hamilton operator $\hat{\mathrm{H}}$ is

$$\hat{\mathrm{H}} = \hat{\mathrm{H}}_0 + \hat{\mathrm{H}}_1$$

in which $\hat{\mathrm{H}}_0$ is the atomic Hamiltonian and $\hat{\mathrm{H}}_1 = -e\,\underline{\hat{\mathrm{r}}}\,\vec{E}(t)$ describes the Hamiltonian of the radiation field. The atom has two stationary, orthonormal eigenstates of $\hat{\mathrm{H}}_0$, $|\,g\,\rangle$ ("ground") and $|\,e\,\rangle$ ("excited") with the eigenvalues ω_g and ω_e respectively. Determine in the following the solution of the time-dependent Schrödinger equation

$$\mathrm{i}\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}|\,\psi(t)\,\rangle = \hat{\mathrm{H}}|\,\psi(t)\,\rangle.$$

a) Show that the Hamilton operator can be represented by the following expression

$$\hat{\mathrm{H}} = \hbar\omega_g|\,g\,\rangle\langle\,g\,| + \hbar\omega_e|\,e\,\rangle\langle\,e\,| + E_0\cos\omega t\left[d_{g,e}|\,g\,\rangle\langle\,e\,| + d_{e,g}|\,e\,\rangle\langle\,g\,|\right]$$

with the dipole moments $d_{i,j} = -e\,\langle i|\vec{\varepsilon}\cdot\underline{\hat{\mathrm{r}}}|j\rangle$.
Hint: The basis $\{|\,g\,\rangle, |\,e\,\rangle\}$ is complete.

b) Use the following *ansatz* for the wavefunction of the system

$$\Psi(\vec{r},t) = c_g(t)\,\Phi_g(\vec{r})\,e^{i\omega_g t} + c_e(t)\,\Phi_e(\vec{r})\,e^{i\omega_e t}$$
$$|\,\Psi(t)\,\rangle = C_g(t)\,|\,g\,\rangle + C_e(t)\,|\,e\,\rangle.$$

Show that the equations for the coefficients result in

$$\mathrm{i}\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}C_g(t) = \hbar\omega_g C_g + E_0\cos(\omega t)\,d_{g,e}\,C_e$$
$$\mathrm{i}\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}C_e(t) = \hbar\omega_g C_e + E_0\cos(\omega t)\,d_{e,g}\,C_g.$$

c) Now, introduce the coefficients

$$c_e(t) = C_e(t)\,\mathrm{e}^{\mathrm{i}\omega_e t}$$
$$c_g(t) = C_g(t)\,\mathrm{e}^{\mathrm{i}\omega_g t}$$

as an *ansatz* to show that

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} c_g(t) = \frac{E_0}{2} d_{g,e} c_e(t) \left[e^{\mathrm{i}(-\omega_{e,g}+\omega)t} + e^{-\mathrm{i}(\omega_{e,g}+\omega)t} \right]$$
$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} c_e(t) = \frac{E_0}{2} d_{e,g} c_g(t) \left[e^{\mathrm{i}(\omega_{e,g}+\omega)t} + e^{\mathrm{i}(\omega_{e,g}-\omega)t} \right].$$

Here: $\omega_{e,g} = \omega_e - \omega_g$.

d) Transform to a rotating coordinate system by introducing the amplitudes

$$\tilde{c}_g(t) = e^{-\mathrm{i}\frac{\delta t}{2}} c_g(t)$$
$$\tilde{c}_e(t) = e^{+\mathrm{i}\frac{\delta t}{2}} c_e(t),$$

where the *detuning* δ is given by $\delta = \omega - \omega_{e,g}$. The commonly used *rotating-wave approximation* (RWA) neglects the faster oscillating terms with $\omega_{e,g} + \omega$ compared to $\omega_{e,g} - \omega$. Define the *Rabi frequency* $\Omega_0 = \frac{d_{e,g} E_0}{\hbar}$ and use the RWA to show that

$$\frac{d}{dt} \begin{pmatrix} \tilde{c}_g(t) \\ \tilde{c}_e(t) \end{pmatrix} \approx -\frac{\mathrm{i}}{2} \begin{pmatrix} \delta & \Omega_0 \\ \Omega_0 & -\delta \end{pmatrix} \begin{pmatrix} \tilde{c}_g(t) \\ \tilde{c}_e(t) \end{pmatrix}.$$

e) Show that the system of equations decouples for the second derivates,

$$\frac{d^2}{dt^2} \begin{pmatrix} \tilde{c}_g(t) \\ \tilde{c}_e(t) \end{pmatrix} \approx -\frac{1}{4} \begin{pmatrix} \delta^2 + \Omega_0^2 & 0 \\ 0 & \delta^2 + \Omega_0^2 \end{pmatrix} \begin{pmatrix} \tilde{c}_g(t) \\ \tilde{c}_e(t) \end{pmatrix}.$$

f) Show that the solutions of these differential equations are given by

$$\tilde{c}_g(t) \approx -\mathrm{i}\frac{\Omega_0}{\Omega} \sin\left(\frac{\Omega t}{2}\right) \tilde{c}_e(0) + \left[\cos\left(\frac{\Omega t}{2}\right) - \mathrm{i}\frac{\delta}{\Omega} \sin\left(\frac{\Omega t}{2}\right)\right] \tilde{c}_g(0)$$
$$\tilde{c}_e(t) \approx -\mathrm{i}\frac{\Omega_0}{\Omega} \sin\left(\frac{\Omega t}{2}\right) \tilde{c}_g(0) + \left[\cos\left(\frac{\Omega t}{2}\right) + \mathrm{i}\frac{\delta}{\Omega} \sin\left(\frac{\Omega t}{2}\right)\right] \tilde{c}_e(0)$$

with the *generalized Rabi frequencies* $\Omega = \sqrt{\delta^2 + \Omega_0^2}$.

g) What are the consequences of the result in f) for the time-dependent population $P_e = |\tilde{c}_e(t)|^2$ of the excited state when the system is at the beginning in its ground state
($c_g(t = 0) = 1$, $c_e(t = 0) = 0$) ?

Exercise 2

Consider an atom interacting with an oscillating light field with constant amplitude and frequency ω. The dipole approximation is used for describing the atom-light interaction in the semiclassical treatment. Using pertubation theory the transition probability can be expressed in the form

$$P(t) \propto \frac{\sin^2\left(\frac{1}{2}(\omega_{m,k} - \omega)t\right)}{(\omega_{m,k} - \omega)^2} \tag{1}$$

where $\omega_{m,k} = \omega_m - \omega_k$ is the transition frequency between the states φ_m and φ_k which are the eigenfunctions of the unperturbed Hamiltonian. The behaviour of this function shall be investigated in the following.

a) Plot function (1) over the detuning $\delta = \omega_{m,k} - \omega$ for (i) $t = 0.5\,\mathrm{s}$, (ii) $t = 1\,\mathrm{s}$, (iii) $t = 5\,\mathrm{s}$, (iv) $t = 10\,\mathrm{s}$ using a plotting program of your choice and explain the behaviour.

b) For nearly resonant light pulses ($\omega \approx \omega_{m,k}$) the model of an effective two-level atom is appropriate. Why?

c) Consider the hdyrogen atom interacting with a laser pulse with a wavelength of 630 nm. Which transitions in a hydrogen atom can be excited with a 1 fs laser pulse?
Hint: Balmer series!!!