The Basics of Lorentz Group Representations

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Markus Schulze

The Lorentz Group & its Representations

The Lorentz group is obviously very important to classical and quantum physics.

The question to be illuminated here is: How do we find the basic building blocks for Lorentz-inv. QFT Lagrangians?

Very familiar: spin-1 particles: $A^\mu(x)$ 4-vector

spin-1/2 particles: $\psi(x)$ 4-spinor

Q: • why does A^μ with $\mu=0,1,2,3$ describe spin-1 with $\lambda = 0,+,-$?
• why isn't $\psi(x)$ a 4-vector like $\psi^\mu(x)$?
• how would you describe a spin-3/2 particle or spin-2?

I.1. Recap: Groups

A group G is a set of elements $G=\{g_i\}$ with a group product $g_1 \circ g_2 = g_3 \in G$ that fulfills

a.) $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$,

b.) $\exists\, e$ with $g_i \circ e = e \circ g_i = g_i$,

c.) $\exists\, g_i^{-1}$ with $g_i \circ g_i^{-1} = e$.

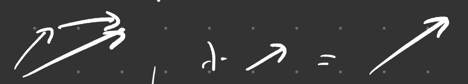
Typical properties: finite/infinite
discrete/continuous
abelian/non-abelian

examples: permutation, rotations, integers under addition $(\mathbb{Z},+)$

I.2. Recap: Group Representations

$\mathbb{R}, \mathbb{C}$ (Körper)

Remember: Vector Space over field $\mathbb{F}$



+ scalar prod., metric, linear operators
basis
Hilbert space

examples: $\mathbb{R}^n$, polynomials with coeff. $\mathbb{F}$

Def: A representation R of a group G on a vector space V is a map from the group elements onto linear operators over V:

so far undefined

$$R: G \mapsto GL(V), \quad D_R(g_i) \mapsto \hat{O}_i$$

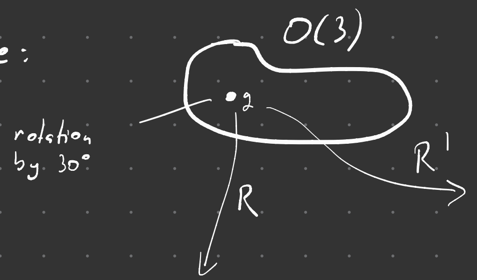
with $D_R(g_1 \circ g_2) = D_R(g_1) \cdot D_R(g_2)$ and $D_R(e) = \mathbb{1}$

Note: • V with scalar prod. can be described by a basis $\{\hat{e}_i\}$ $\quad v_i = c_1 \hat{e}_1 + \dots + c_n \hat{e}_n$

• Linear operators $D_R(g_i)$ depend on the basis

• for one and the same group G, the dimension of V and of D_R can vary.

example:



$D_{R'}(g) = \begin{pmatrix} \vdots & \vdots & \vdots & 0 \\ & & & \\ 0 & & & 1 \end{pmatrix}$

$D_R(g) = \begin{pmatrix} \vdots & \vdots & \vdots \end{pmatrix}$

$(\vdots)$ in V acts on vectors/states

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4min.

I.3. Recap: Lie Groups & Lie Algebras

Familiar notions: $SU(N)$, generators, structure constants...

Def.: A Lie group is a group, where $g_1 \circ g_2$ and $g \circ g^{-1}$ are smooth maps.

↖ can take derivatives
smooth manifold

Representation of a Lie group element can be written as $D_R(g_i) = \exp(\theta_i X_R^i)$ with $\theta_i \in \mathbb{R}$ and the X_R^i are group generators of the corresponding Lie algebra.

In this sense: $g_i \hat{=} g_i(\theta^a)$ can be differentiated wrt. θ^a.

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4min.

Def.: Lie algebra $\mathfrak{g} = (V, [\cdot,\cdot])$ where $[\cdot,\cdot] : V \times V \to V$ and $[X,X] = 0$ and Jacobi-Id.

$$[X,[Y,Z]] + \dots + \dots = 0.$$

Lie algebra of $SU(3)$

example: $SU(3)$: $D_R(g_i) = U = \exp(i\,\theta_a t_R^a)$, $t_R^a \in su(3)$

where $[t_R^a, t_R^b] = i f^{abc} t_R^c$

3 dim. R: fundam. repr. $(t_F^a) = (\cdot\cdot)$

8 dim. R: adjoint repr. $(t_A^a) = (\cdots)$

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II.1. The Lorentz Group

recap: $\Lambda^\mu{}_\nu$ with $x'^\mu = \Lambda^\mu{}_\nu$ that leaves $(x^0)^2 - (\vec{x})^2$ inv.

$D_R(\Lambda) = \Lambda$ $\Lambda \in O(3,1)$

Spans $O(3,1)$, depending on $|\Lambda^0{}_0| \geq 1$ and $\det\Lambda = \pm 1$ there are 4 disconnected patches. Typically choose: proper orthochronous $\det\Lambda = +1$, $\Lambda^0{}_0 \geq 1$: $SO(3,1)^\dagger$.

The other patches can be obtained from $SO(3,1)^\dagger$ by applying P (space inv.) and/or T (time inv.) operations

We actually need $(dx^0)^2 - (d\vec{x})^2$ to be inv. (for const. speed of light in every frame)

$\Rightarrow$ Poincare trafos. (not discussed here)

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II.2. Representations

infinit.: $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu + \mathcal{O}(\omega^2)$ where $\omega^{\mu\nu} = -\omega^{\nu\mu}$

$(\omega^{\mu\nu})$ = anti-symm. (4×4) $\Rightarrow$ 6 d.o.f. $\Rightarrow$ 3 rotations, 3 boosts

Lorentz group is a Lie group, hence an $\Lambda \in SO(3,1)$ has the representation $D_R(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\mu\nu} J_R^{\mu\nu}\right)$.

$(J_R^{\mu\nu})_{ij}$ are 6 $(n\times n)$ matrices (R defines value of n) that fulfill Jacobi-Id. and $i[J_R^{\mu\nu}, J_R^{\rho\sigma}] = g^{\mu\sigma}J^{\nu\rho} + g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho}$ (*)

Then any "state" $\phi_i' = \exp\left(-\frac{i}{2}\omega_{\mu\nu}J_R^{\mu\nu}\right)_{ij}\phi_j$ given a basis for ϕ_i in V, which is labeled by R.

well-known examples: a.) scalar repr. ϕ_i with $i=1$ then

$(J^{\mu\nu}_{scalar})_{ij}$ = one number, $\phi' = e^{i\varphi}\phi$, choose $(J^{\mu\nu}_{scalar})_{ij} = 0$

b.) 4-vector repr.: $\phi_i \hat{=} A_\mu$ $\quad i = 1,\dots,4,$ or $\mu = 0\dots 3$

$$A'^\alpha = \underbrace{\exp\left(-\tfrac{i}{2}\omega_{\mu\nu} J^{\mu\nu}_{4vec}\right)^\alpha{}_\beta}_{\Lambda^\alpha{}_\beta} A^\beta$$

works for $(J^{\mu\nu}_{4vec})^\alpha{}_\beta = -i\left(g^{\mu\alpha}\delta^\nu_\beta - g^{\nu\alpha}\delta^\mu_\beta\right)$

c.) spinor repr.: $\phi_i \hat{=} \psi_i$ with $i = 1,2,3,4$

... $\Rightarrow$ $(J^{\mu\nu}_{spinor})_{ij} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]_{ij}$

Q: Is there a system. way to obtain these repr.? What the relation to spin? Can it be generalized?

II. 3. General Irreducible Representations

We know that $J^{\mu\nu}$ is asymm. w.r.t. $\mu\nu$.

$\Rightarrow$ decompose into $K^i := J^{i0}$, $J^i := \frac{1}{2}\varepsilon^{ijk}J^{jk}$

$$(J^{\mu\nu}) = \begin{pmatrix} 0 & & \\ \boxed{K} & \boxed{J} & \\ & & 0 \end{pmatrix}$$

then ⊛ $[J^{\mu\nu}, J^{\rho\sigma}] = \dots$

becomes

$[J^i, J^j] = i\varepsilon^{ijk}J^k$ $\Leftarrow$ defines angular mom. / spin

$[J^i, K^j] = i\varepsilon^{ijk}K^k$
$[K^i, K^j] = -i\varepsilon^{ijk}J^k$ $\Big\}\Leftarrow$ characteristic for spatial vector

$$\Rightarrow \quad D_R(\lambda) = \exp\left(-i\vec{\theta}\cdot\vec{J}_R + i\vec{\eta}\cdot\vec{K}_R\right)$$

with $\theta_i = \frac{1}{2}\varepsilon^{ijk}\omega^{jk}$ rotation angle
$\eta_i = \omega^{i0}$ boost parameter

This allows a very intuitive repr. in terms of rotations & boosts.

Let's decompose this a second time: $\begin{cases} A^i = \frac{1}{2}(J^i + iK^i) \\ B^i = \frac{1}{2}(J^i - iK^i) \end{cases}$

$$\Rightarrow D_R(\lambda) = \exp(i\vec{\alpha}\cdot\vec{A})\cdot\exp(i\vec{\beta}\cdot\vec{B}) \qquad \begin{matrix}\vec{\alpha} = -i\vec{\theta} + \vec{\eta} \\ \vec{\beta} = -i\vec{\theta} - \vec{\eta}\end{matrix}$$

no extra term because of Baker-Campbell-Hausdorff

with $[A^i, A^j] = i\varepsilon^{ijk}A^k$
$[B^i, B^j] = i\varepsilon^{ijk}B^k$
$[A^i, B^j] = 0$
$\Big\}$ two copies of angular mom. algebra that are not coupled

Since $J^i = A^i + B^i \Leftrightarrow \left(\boxed{n\times n}\right) = \begin{pmatrix}\boxed{A_i} & 0 \\ 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \boxed{B_i}\end{pmatrix} = A_i \oplus B_i$

we label rows/columns of A_i, B_i with $m_a = -a, -a+1, \dots, +a$
$m_b = -b, -b+1, \dots, b$

and reorganize states $\phi = \left(\boxed{n}\right) \rightarrow \phi_A \otimes \phi_B \Leftrightarrow \left(\boxed{2a+1}\right) \otimes \left(\boxed{2b+1}\right) = \left(\boxed{(2a+1)\times(2b+1)}\right)$
$n = (2a+1)(2b+1)$

then $D_R(\lambda) = \exp(i\vec{\alpha}\cdot\vec{A}) \otimes \exp(i\vec{\beta}\cdot\vec{B}) =: D_{(a,b)}(\lambda)$

such that $D_{(a,b)}(\lambda)(\phi_A \otimes \phi_B) = \exp(i\vec{\alpha}\cdot\vec{A})\,\phi_A \otimes \exp(i\vec{\beta}\cdot\vec{B})\,\phi_B$

Since $\vec{A}, \vec{B}$ are two uncoupled spins and $\vec{J} = \vec{A} \oplus \vec{B}$, the usual spin addition rules apply.

$\Rightarrow$ A state that transforms according to $D_{(a,b)}$ has components that rotate like spin $j = |a-b|, \dots, a+b$.