Now, we made the connection between spin of a state and its representation of the proper orthochr. Lorentz group.

E.g.: to describe spin $j = 5/2$ we need e.g. $a = 2$, $b = 1/2$.
The corresp. state will have components that describe spin $3/2$ $(= |a-b|)$ and spin $5/2$ $(= a+b)$.

The basic building blocks to construct any repres. for spin j states are the spin-1/2 repres.

To understand this statement better, let's look at the $su(2)$ spin algebra.

Intermezzo: Representations of $SU(2)$

Lie algebra $su(2)$ of $SU(2)$: $[s^i_R, s^j_R] = i\varepsilon^{ijk} s^k_R$

e.g. 2-dim. repr.: $s^i = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$ acting on $\psi = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$

3-dim. repr.: $s^i = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ acting on $\psi = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$

From QM we remember that 2-dim. corresp. to spin-1/2 with $m = \pm 1/2$ and 3-dim. corresp. to spin-1 with $m = +1, 0, -1$. These states are often written as $|j, m\rangle$. The dim. of the repr. is the size of $\{m\} = \{-j, \dots, +j\}$.

Q: How do we explicitely construct the s^i matrices for a given spin?

$\Rightarrow$ There's a recipe! (The recipe doesn't involve using $\vec{s}^2$ or the property that $[\vec{s}^2, s_i] = 0$ in contrast to typical QM derivations)

Step ①: None of the three s_i commute with another. But one can find a unitary $U^\dagger U = 1$ that transforms $\psi_i' = U\psi_i$, $s_i' = U^\dagger s_i U$ such that one of the s_i becomes diagonal. Choose: s_3'.

Step ②: Pick the states $|s, m\rangle$ that are eigenstates to the diagonal s_3.

Step ③: Define ladder operators $s_\pm = \frac{1}{\sqrt{2}}(s_1 \pm i s_2)$ satisfying

$$[s_3, s_\pm] = \pm s_\pm \quad \text{and} \quad [s_+, s_-] = s_3.$$

Since $s_3 |s, m\rangle = m\, |s, m\rangle$,

$$s_3 \left(s_\pm |s, m\rangle\right) = s_\pm s_3 |s, m\rangle \pm s_\pm |s, m\rangle = (m \pm 1)\, s_\pm |s, m\rangle$$

$$\overset{\text{cf. } s_3}{\Longrightarrow} \quad s_\pm |s, m\rangle \sim |s, m \pm 1\rangle \quad \leftrightarrow \quad s_\pm |s, m\rangle = N_\pm |s, m \pm 1\rangle$$

with $N_\pm = \frac{1}{\sqrt{2}} \sqrt{(s \mp m)(s + 1 \pm m)}$, $m \in \{-s, \dots, s\}$, $\langle s, m | s, m'\rangle = \delta_{mm'}$.

Step ④: Now, we can construct the actual matrix elements of the (s_i):

$$(s_i)_{m'm} = \langle s\, m' | s_i | s\, m\rangle = \begin{pmatrix} \vdots & \cdots \end{pmatrix}_{m'm}$$

$$\Rightarrow \quad \langle s\, m' | s_3 | s\, m\rangle = m\, \delta_{mm'}$$

$$\langle s\, m' | s_1 | s\, m\rangle = \frac{1}{\sqrt{2}} \langle s\, m' | s_+ + s_- | s\, m\rangle = \frac{1}{\sqrt{2}}\left(N_+ \delta_{m'(m+1)} + N_- \delta_{m'(m-1)}\right)$$

$$\langle s\, m' | s_2 | s\, m\rangle = \frac{i}{\sqrt{2}} \langle s\, m' | s_- - s_+ | s\, m\rangle = \frac{i}{\sqrt{2}}\left(N_- \delta_{m'(m-1)} - N_+ \delta_{m(m+1)}\right)$$

Done!

17

Examples: a.) $s = 1/2 \Rightarrow m \in \{-1/2, +1/2\} \Rightarrow$ 2-dim.

$$\mathcal{N}_+(m=-1/2) = \mathcal{N}_-(m=+1/2) = 1, \quad \mathcal{N}_+(m=+1/2) = \mathcal{N}_-(m=-1/2) = 0$$

$$(S_1) = \left(\tfrac{1}{2}\,\delta_{m'(m-1)}\right) = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

$$(S_2) = \left(\tfrac{i}{2}\,\delta_{m'(m-1)}\right) = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}$$

$$(S_3) = (m\,\delta_{m'm}) = \begin{pmatrix} -1/2 & 0 \\ 0 & +1/2 \end{pmatrix}$$

note: $\vec{s} = \frac{1}{2}\underset{\text{Pauli}}{\vec{\sigma}}$

b.) $s = +1 \Rightarrow m \in \{-1, 0, +1\} \Rightarrow$ 3-dim. $(\ldots)$

$$(S_1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (S_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad (S_3) = \begin{pmatrix} +1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

18

Reduction of Direct Product Representations

recap:

a.) Direct sum: $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \mathrm{diag}(A, B)$

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & & \end{pmatrix} \oplus \begin{pmatrix} \vdots & \vdots \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & & \end{pmatrix} & \\ & \begin{pmatrix} \vdots & \vdots \end{pmatrix} \end{pmatrix}$$

If an arbitr. matrix can be written as $M = \mathrm{diag}(M_1, \ldots, M_n)$ then it is reducible. Otherwise it is irreducible.

b.) Direct product: $A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} \xrightarrow[\text{reduc.}]{\text{if}} C_1 \oplus \ldots \oplus C_m$

some unitary transformation

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & & \end{pmatrix} \otimes \begin{pmatrix} \vdots & \vdots \end{pmatrix} = \begin{pmatrix} \cdot(\vdots) & \cdot & \cdot & \cdot \\ \cdot(\vdots) & & & \\ \cdot(\vdots) & & & \end{pmatrix}$$

if reduc.: $U^{+}(A \otimes B)U = C_1 \oplus \ldots \oplus C_m$

19

To write direct products $A \otimes B$ of $su(2)$ representations in terms of direct sums, we find the transforming unitary matrices U by applying the regular rules for adding angular momentum. The matrix elements of U are the well known Clebsh-Gordon coeff.:

$$|J, M\rangle = \sum_{m_1, m_2} \underbrace{\langle j_1\, j_2\, m_1\, m_2 | J\, M\, ; j_1\, j_2\rangle}_{U} |j_1 m_1\rangle \otimes |j_2 m_2\rangle$$

Hence: $D_{(\text{spin } s_1)} \otimes D_{(\text{spin } s_2)} = \bigoplus_{s=|s_1-s_2|}^{s_1+s_2} D_{(\text{spin } s)}$

example: $D_{(\text{spin } 3/2)} \otimes D_{(\text{spin } 1)} = D_{(\text{spin } 1/2)} \oplus D_{\text{spin}(3/2)} \oplus D_{(\text{spin } 5/2)}$

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \vdots & & & \end{pmatrix} \otimes \begin{pmatrix} \cdot & \cdot & \cdot \\ \vdots & & \end{pmatrix} = \begin{pmatrix} 12 \times 12 \end{pmatrix} = \begin{pmatrix} \boxed{2\times2} & & \\ & \boxed{4\times4} & \\ & & \boxed{6\times6} \end{pmatrix}$$

d.o.f. of states: 4 d.o.f. $\times$ 3 d.o.f. $=$ 12 d.o.f. $=$ 2+4+6 d.o.f.

20

End of Intermezzo

Now, we have all necessary ingredients:

We can explicitely construct spin j representations of $\vec{A}$ and $\vec{B}$. And we can use the direct product to obtain higher spin repr. from lower spin repres.

- $D_{(0,0)}$: $a=b=0$ represents spin $j=0$ states with 1-dim. repres. with $\vec{A}=\vec{B}=0 \Rightarrow D_{(0,0)}=1 \Rightarrow \phi'=\phi$
- $D_{(1/2,0)}$: gives spin $j=1/2$ states with 2-dim. repres. $\vec{A}=\frac{1}{2}\vec{\sigma}$ (see p. 17), $\vec{B}=0 \Rightarrow D_{(1/2,0)}=\exp\left(+\frac{i}{2}\vec{\alpha}\cdot\vec{\sigma}\right)$
  χ_i = 2 component left-handed Weyl spinor and anti-spinor.
- $D_{(0,1/2)}$: same as above for right-handed Weyl spinor

- $D_{(1/2,0)} \oplus D_{(0,1/2)}$: combination of 2 comp. LH+RH Weyl spinors $\Rightarrow$ 4 dim. repr.
  $$\begin{pmatrix} D_{(1/2,0)} & 0 \\ 0 & D_{(0,1/2)} \end{pmatrix}$$
  $\Rightarrow$ Dirac spinor $\psi=\begin{pmatrix}\chi_L\\ \chi_R\end{pmatrix}$
  4 d.o.f. for $m=\pm 1/2$ for particle and anti-particle.
- $D_{(1/2,1/2)}$: $a=b=1/2$ gives states with spin $j=0,1$
  $$D_{(1/2,1/2)} = \underset{(\because)}{\exp\left(\tfrac{i}{2}\vec{\alpha}\cdot\vec{\sigma}\right)} \otimes \underset{(\because)}{\exp\left(\tfrac{i}{2}\vec{\beta}\cdot\vec{\sigma}\right)} = \underset{(\Lambda^\mu{}_\nu)}{\left(\vdots\right)} \Rightarrow \text{4-dim. represent.}$$
  $\Rightarrow$ states are 4-vectors A^μ.
- $\bigotimes_{n=1}^{N} D_{(1/2,1/2)} = \bigoplus_{k,k'=0}^{N/2} D_{(k,k')}$ gives states that transform like N 4-vectors $\Rightarrow$ rank N tensor
  spin $j=k+k'$ is always integer

- states with j = half integer are obtained from
  $$\bigotimes_{n=1}^{N} D_{(1/2,1/2)} \otimes \left(D_{(1/2,0)} \oplus D_{(0,1/2)}\right)$$
  i.e. tensor $\otimes$ Dirac spinor

e.g. $j=\frac{3}{2}$: $\left(\frac{1}{2},\frac{1}{2}\right)\otimes\left(\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right)=\left(0,\frac{1}{2}\right)\oplus\left(1,\frac{1}{2}\right)\oplus\left(\frac{1}{2},0\right)\oplus\left(\frac{1}{2},1\right)$

the corresp. state is a spinor-4-vector $\psi^\mu=\begin{pmatrix}\psi_1^\mu\\ \vdots\\ \psi_4^\mu\end{pmatrix}$.

<u>Note:</u> So far we only considered proper orthochr. Lorentz group $SO(3,1)^\uparrow$. In general transf. there is an element $P: \vec{x}\to-\vec{x}$ with
$$P\vec{J}P^{-1}=\vec{J} \qquad\qquad P\vec{A}P^{-1}=\vec{B}$$
$$P\vec{K}P^{-1}=-\vec{K} \quad \longleftrightarrow \quad P\vec{B}P^{-1}=\vec{A}$$
Hence, a repr. $D_{(a,b)}$ of $SO(3,1)^\uparrow$ is not a repr. of the general Lorentz group.

$\Rightarrow$ We need to use $D_{(a,b)}\oplus D_{(b,a)}$ if $a\neq b$!

- One example is $D_{(1/2,0)}\oplus D_{(0,1/2)}$ which we encountered already for Dirac spinors
- Another example: $D_{(1,0)}\oplus D_{(0,1)} \Rightarrow$ spin $j=1$
  $\left(\because\right)\oplus\left(\because\right) \Rightarrow 6$ d.o.f.
  transforms a rank 2 tensor but only 6 d.o.f.
  $\Rightarrow$ anti-symm. tensor $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$
- let's go back to general rank 2 tensors:
  $$\underbrace{D_{(1/2,1/2)}\otimes D_{(1/2,1/2)}}_{4\times4\text{ d.o.f.}}=\underbrace{D_{(1,1)}}_{3\times3\text{ d.o.f.}}\oplus\underbrace{D_{(0,1)}\oplus D_{(1,0)}}_{6\text{ d.o.f.}}\oplus\underbrace{D_{(0,0)}}_{1\text{ d.o.f.}}$$
  asym. trace

$\Rightarrow D_{(1,1)}$ transforms a spin $j=2$ state with 9 d.o.f.
= symmetric traceless tensor

II. 3. Physical d.o.f

1.) Dirac spinors $D_{(1/2,0)} \oplus D_{(0,1/2)}$ with $\psi \to 4$ d.o.f. ✓
describes LH and RH spin $1/2$ particle + antiparticle.

2.) Massive 4-vector $D_{(1/2,1/2)}$ with $(A_\mu) \to 4$ d.o.f.
$j = 0, 1$

A_0 : spin 0
$\vec{A}$: spin 1

the spin-0 d.o.f. in A_0 doesn't lead to a physical field excitation because one requires $\partial_\mu A^\mu = 0$ in all theories.

3.) Massive Rarita-Schwinger field ψ^μ describes spin $3/2$ states but has $4 \times 4 = 16$ d.o.f.

repr.: $D_{(1,1/2)} \oplus D_{(1/2,1)} \oplus D_{(1/2,0)} \oplus D_{(0,1/2)}$

One requires the gauge cond. $\psi^\mu \gamma_\mu = 0$, this removes the pure spinor parts $D_{(1/2,0)} \oplus D_{(0,1/2)}$ $= 4$ d.o.f. and one requires $\partial_\mu \psi^\mu = 0$, which removes another 4 d.o.f.

$\Rightarrow$ 8 d.o.f. remaining that describe spin-$3/2$ particle and antiparticle.

4.) Massive spin-2 field : $D_{(1,1)}$ with $H_{\mu\nu}$
symm. traceless tensor $\hat{=}$ 9 d.o.f.
Phys. theories where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ need

a gauge condition $\partial_\mu H^{\mu\nu} = 0$, where $H_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h$
(Hilbert gauge) removes 4 d.o.f.

$\Rightarrow$ 5 d.o.f. remain in $H^{\mu\nu}$ and describe spin 2 state.

5.) discussion of massless spin 1, 2 states is tricky and requires discussion of the general Poincaré group (little group: translations $\Leftrightarrow$ mass label).

NOT DONE HERE.

$\boxed{\text{FIN}}$

Literature:
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