Fundamentals of Optical Sciences

WS 2015/2016 **11 . Exercise** **11.01.2016**

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Prepare your answers for the exercise on 18.01.2016.

Exercise 1
For a 1D harmonic oscillator with mass m and frequency ω, consider the eigenstates $|\,\alpha\,\rangle$ of the annihilation operator â, $\hat{a}\,|\,\alpha\,\rangle = \alpha\,|\,\alpha\,\rangle$ with complex eigenvalues α. The eigenstates $|\,\alpha\,\rangle$ should be expanded in the basis $\{\,|\,n\,\rangle\}$ of the energy and number operator eigenstates, so that $|\,\alpha\,\rangle = \sum_n c_n\,|\,n\,\rangle$.

a) Determine the eigenstates $|\,\alpha\,\rangle$ by

   i) identifying the recursion relation of the coefficients c_n,
   ii) expressing the coefficients c_n as a function of c_0,
   iii) identifying the coefficient c_0 using the normalization condition $\langle\alpha|\alpha\rangle = 1$, and
   iv) calculating the probability of measuring the energy eigenvalue $E_n = \hbar\omega\,(n + 1/2)$ of the state $|\,\alpha\,\rangle$.

b) Characterize the states $|\,\alpha\,\rangle$ concerning their energy by calculating

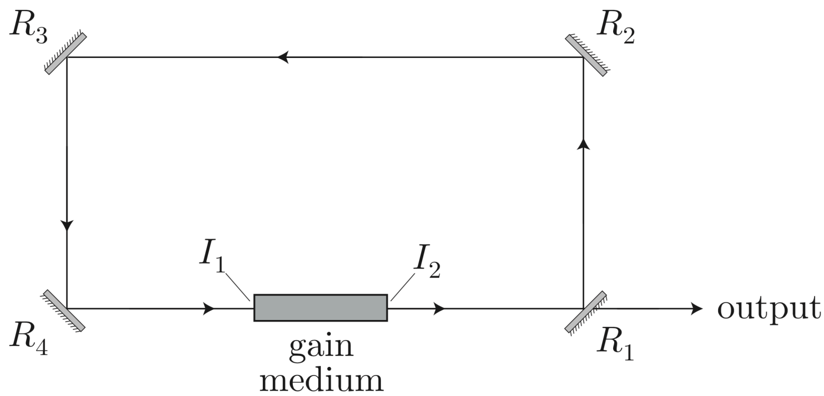
   i) the expectation value of the energy $\langle\,E\,\rangle$,
   ii) the expectation value of the squared energy $\langle\,E^2\,\rangle$,
   iii) the resulting variance $\Delta E^2 = \langle\,E^2\,\rangle - \langle\,E\,\rangle^2$, uncertainty $\Delta E = \sqrt{\Delta E^2}$, and relative uncertainty $\Delta E\,/\,\langle\,E\,\rangle$.
   iv) In which sense does the energy become better defined with increasing $|\alpha|$?

c) Characterize the states $|\,\alpha\,\rangle$ concerning their expectation values of position and momentum by calculating

   i) the position expectation value $\langle\,q\,\rangle$,
   ii) the momentum expectation value $\langle\,p\,\rangle$, and
   iii) the product of the position and momentum uncertainties $\Delta q \cdot \Delta p$.

Exercise 2

Consider the ring cavity laser illustrated here, where I_1 and I_2 indicate the intensities before and after the gain medium as shown, and the gain medium has length l_g.



Since the cavity has a ring configuration, light passes through the gain medium in one direction only. Starting with the single-pass gain relation for a medium of length l_g,

$$\log G + \frac{I_1}{I_{\mathrm{sat}}}(G-1) = \gamma_0\, l_g \tag{1}$$

where $G = I_2/I_1$ is the gain and γ_0 is the small-signal gain coefficient, show that the optimum output transmission for mirror R_1 is

$$T_1(\text{optimal}) \approx \sqrt{\gamma_0 l_g P_l} - P_l \tag{2}$$

where P_l is the probability of *loss* in the other 3 mirrors.

Assume the laser cavity is operating in steady state (so that $G = 1/P_s$ where P_s is the total survival probability), and that all losses are due to mirror reflectivity. Further, work in the low-loss regime, where $1 - P_s$ is small, so that

$$\log \frac{1}{P_s} = -\log\left[1 + (1 - P_s)\right] \approx 1 - P_s\ . \tag{3}$$