Quantum Information & Quantum Computing

(Experimental Part)

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## Contents

1 Introduction 5

2 Optical and Cavity QED Implementations 7
   2.1 Properties of an optical quantum computer 7
   2.1.1 Single Quantum Gates for Photons 7
   2.1.2 Photon-Photon-Interaction 9
   2.2 Quantum Computation with Linear Optics 11
   2.2.1 Hong-Ou-Mandel Interference 11
   2.2.2 The KLM-Gate 13
   2.2.3 Another intuitive two-photon gate 15
   2.2.4 Experimental demonstrations 16
   2.2.5 Scalability of LOQC 17
   2.2.6 Summary: Optical photon quantum computer 19
   2.3 Cavity Quantum electrodynamics 20
   2.3.1 Jaynes-Cummings-Hamiltonian 20
   2.3.2 Resonant Interaction 22
   2.3.3 Off Resonant Interaction/Phase Shifts 27
   2.3.4 Flying Qubits 30
   2.3.5 Summary: Cavity QED implementation of quantum computers 33

3 Ion Trap Implementations 34
   3.1 Trapping Ions 34
   3.1.1 Paul traps 34
   3.1.2 Trapping ion strings 37
   3.1.3 Normal Modes 40
   3.2 Laser Cooling 42
   3.2.1 Doppler Cooling 43
   3.2.2 Harmonic potential 44
   3.2.3 Sideband Cooling 44
   3.2.4 Choosing atoms 45
   3.3 Single and Two Qubit Gates 47
   3.3.1 Hamiltonian of ions in a trap 47
   3.3.2 Interaction with a laser field 47
   3.3.3 Single qubit operation 49
   3.3.4 Two qubit operation 50
   3.4 Experimental Realization of a CNOT Gate 51
   3.5 Gates and Tricks with Single Ions 56
   3.5.1 Demonstration of Deutsch-Jozsa 56
   3.5.2 Teleportation of atomic qubits 57
   3.5.3 A Quantum Byte 60
   3.5.4 Novel Trap Designs 62
4 Solid State Implementation of Quantum Computers

4.1 Superconducting Qubits
4.2 Josephson Charge Qubits
  4.2.1 Single qubit gates
  4.2.2 Experimental Results on Single qubit rotation
  4.2.3 Two-qubit gates
4.3 Josephson Flux Qubits
4.4 Readout of Josephson Qubits with SETs
4.5 Silicon-based Quantum Computation/Spintronics
  4.5.1 Single qubit gates
  4.5.2 Two-qubit gates
  4.5.3 Single spin measurement
4.6 Quantum Dots as Qubits
  4.6.1 Fabrication of quantum dots
  4.6.2 Single qubit gates
  4.6.3 Two-qubit gates
  4.6.4 Initialization and read-out
  4.6.5 Optical manipulation of quantum dots
4.7 Summary: Solid State Implementation
1 Introduction

Some requirements for the physical implementation of quantum computation (according to David DiVincenzo, Fortschr. Phys. 48, 9-11, p. 771 (2000)):

1. **A scalable physical system with well characterized qubits.**
   As discussed above a qubit could be encoded in any two different quantum states. Examples are given in the following. The main consideration is whether the step from 2 qubits to \( n \) qubits is possible. This situation may be compared to the first implementations of classical computers:
   The first machines made from electro-mechanical (Zuse Z1 1938) and later electrical subunits (electrical tubes, IBM 1948) could have been scaled up in principle to increase the computer’s power. But, realistically the failure probability and also the energy consumption would have scaled as well. Only the invention of the semiconductor based subunits made scaling feasible.

2. **The ability to initialize the state of the qubits to a simple initial state, such as \( |0,0,...0\rangle \).**
   In almost all of the above described algorithms initialization is a first step. This initialization may be easy for some physical systems, but crucial for others. If the qubit is encoded in two bits, say with an energy difference \( \Delta E \) then initialization can be accomplished by cooling to below \( T << \Delta E/k_b \). This thermal relaxation of the qubits may take too long a time for a computation. Active initialization may be required.

3. **Long relevant decoherence times, much longer than gate operations.**
   Decoherence is crucial for any quantum computation. It transforms pure states in statistical mixtures and destroys entanglement. The problem of decoherence is the more difficult the larger the quantum system gets: *We do not observe quantum objects in our classical world!* The issue that decoherence must be negligible as long as the computation takes was considered the main obstacle in quantum computation. The suggestion of quantum error correcting codes (Shor 1995) demonstrated that this is not the case! Error correcting codes and fault tolerant computation make arbitrary long calculations possible as soon as the decoherence is negligible for about \( 10^4 - 10^5 \) clock times. This is still a very stringent requirement. The price one has to pay is to have each qubit encoded with 3-10 additional qubits for error correction, which then requires even more complex systems.

4. **A universal set of quantum gates.**
   A set of universal gates consists e.g. of single qubit operations and a \( CNOT \) gate. Gates are unitary operators. In some systems no appropriate
gates may be constructed easily. Interaction between qubits may influence not only one or two, but many other qubits. Interactions can not be switched on and off arbitrarily and can not be controlled with arbitrary precision.

5. A qubit-specific measurement capability.
It is not easy in some systems to perform a measurement with 100% precision. However, this is less of a problem, since the computation may be repeated several times. (If a detection works only with 90%, then a three times repetition already gives 97% fidelity).

For quantum computation the first 5 requirements suffice, but in a wider context of quantum information processing or networking two more features would be desirable:

1. The ability to interconvert stationary and flying qubits.

2. The ability to faithfully transmit flying qubits between specific locations.

The notation "flying" qubits was introduced by Kimble. It refers to photons which should be the ideal long-distance carriers for quantum information. The use of the well developed optical fiber communication system would be a tremendous advantage. However, transport (although over relatively short distances) is also discussed in some solid-state implementation of quantum computers.

Figure 1: Some details of the Zuse Z1 (Berlin Technical Museum)

The above mentioned items will be discussed from the viewpoint of the different systems introduced in the following section. To summarize: There is no fundamental physical problem to build an arbitrary large quantum computing device.
2 Optical and Cavity QED Implementations

2.1 Properties of an optical quantum computer

2.1.1 Single Quantum Gates for Photons

Single photons can represent qubits. The advantage of using photons is that they are easy to produce and to detect with high efficiency. A qubit is usually represented with two optical modes:

\[ |1 \rangle = |1 \rangle_1 |0 \rangle_2 = |10 \rangle = \text{one photon in mode one, zero photons in mode two or} \]
\[ |0 \rangle = |0 \rangle_1 |1 \rangle_2 = |01 \rangle = \text{one photon in mode two, zero photons in mode one} \]

This representation of qubits is called the \textit{dual-rail representation}. The free time evolution of the qubits is given by the Hamiltonian:

\[ H = \hbar \omega a^\dagger a \]

where \( a^\dagger \) and \( a \) are the creation and annihilation operators for a photon. The free evolution thus only adds an overall phase which can be neglected.

The modes can be two physically separated modes (two "beams") or two modes with orthogonal polarization (horizontal and vertical). Single qubit gates can very easily be realized with the help of linear optical elements (mirrors, phase shifters, beam splitters).

Two ways to encode a qubit (two polarizations or two spatial modes) can be easily exchanged.

Figure 2: Transfer of polarization encoding into encoding via two spatial modes and vice versa [Myers and Laflamme, arXiv:quant-ph/0512104 v1 13 Dec 2005]
A mirror is used to redirect beams and arrange networks. Low loss mirrors with 99.9% reflectivity are obtainable.

A phase shifter is a slab of transparent (low loss) material with index of refraction $n$. It simply retards one optical mode with respect to the other. Its action is described as:

$$R_z(\Delta) = \exp(-i\sigma_3 \Delta/2)$$

with the Pauli matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\Delta = (n-n_0)L/c_0$

The phase shift $\Delta$ is defined by the different optical paths through the material of length $L$. The following gives a graphical representation of the phase gate where the upper wire corresponds to the state $|0\rangle = |01\rangle$ and the lower wire to the state $|1\rangle = |10\rangle$:

![Figure 3: Graphical representation of a phase gate](image)

A beam splitter is a piece of transparent (low loss) material with a thin metal coating.

A beam splitter acts on two modes which can be described by annihilation and creation operators $a, a^\dagger$ and $b, b^\dagger$, respectively. There is a transformation between creation/annihilation operators before (subscript 0) and after (subscript 1) a beam splitter with a reflectivity of $R = \cos^2 \theta$ as follows (e.g., for the annihilation operators):

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

Thus, with e.g. the state $|\Psi_{in}\rangle = |0\rangle = |01\rangle = b_1^\dagger |00\rangle$ as input the output is

$$|\Psi_{out}\rangle = (\sin \vartheta a_1 + \cos \vartheta b_1^\dagger) |00\rangle = \cos \vartheta |01\rangle + \sin \vartheta |10\rangle$$
or with the state $|\Psi_{in}\rangle = |1\rangle = |10\rangle = a_0^\dagger |00\rangle$ as input the output is

$$|\Psi_{out}\rangle = (\cos \vartheta a_1 - \sin \vartheta b_1) |00\rangle = \cos \vartheta |10\rangle - \sin \vartheta |01\rangle$$

The beam splitter together with a $\pi$-phase shifter acts (up to an overall phase) as a Hadamard gate.

Phase shifters and beamsplitters thus allow the construction of arbitrary single qubit gates!

### 2.1.2 Photon-Photon-Interaction

The main obstacle of optical quantum computation is to realize controlled gates which require a qubit-qubit (therefore a photon-photon) interaction. Interaction between photons can be established inside certain non-linear materials. Non-linear effects are rather weak. Strongest non-linearities are usually achieved close to resonances which then means a pronounced absorption as well. One type of non-linear material leads to the **Kerr-Effect**. The Kerr-Effect is described by the following Hamiltonian:

$$H_{Kerr} = -\chi a^\dagger a b^\dagger b$$

After propagation through some Kerr material of length $L$ it corresponds to a transformation $K$:

$$K = \exp(i\chi L a^\dagger a b^\dagger b)$$
$K$ acts on the basis states $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$ as follows:

\[
\begin{align*}
K|00\rangle &= |00\rangle \\
K|01\rangle &= |01\rangle \\
K|10\rangle &= |10\rangle \\
K|11\rangle &= \exp(i\chi L)|11\rangle
\end{align*}
\]

If a huge Kerr effect is achieved with $i\chi L = i\pi$ and thus $K|11\rangle = -|11\rangle$ then a CNOT can be constructed, because CNOT can be factorized as:

\[
U_{\text{CNOT}} = (I \otimes H)K(I \otimes H)
\]

where

\[
(I \otimes H) = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & -1
\end{pmatrix}
\text{ and } K = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

The following figure shows a possible realization of a Kerr-gate using dual-rail representations of single photon qubits: The presence of light (single photon) in mode $s$ produces a phase shift in one interferometer arm, thus sending a photon to detector $D_1$ and $D_2$, respectively.

![Figure 5: Schematics of a Kerr-gate](image-url)
2.2 Quantum Computation with Linear Optics

2.2.1 Hong-Ou-Mandel Interference

Recent proposals suggest a strong photon-photon interaction without using non-linear material. These gates rely on two-photon interference of indistinguishable photons. Linear optics quantum computation (LOQC) has the advantage of experimental simplicity. A review can also be found in Kok et al. [Rev. Mod. Phys. 79, 135 (2007)]. Figures 5 and 11-15 are from this review.

As can be seen from Fig. 6 two identical photons impinging on a beam-splitter

![Diagram of two-photon interference on a beam-splitter](image)

Figure 6: Principle of two-photon interference on a beam-splitter for classical particles, bosons, and fermions

will always leave together. This quantum mechanical effect was first observed by Hong, Ou, and Mandel (Phys. Rev. Lett. 59, 2044 (1987)). Two identical photons were created by parametric down-conversion in a non-linear crystal. The experimental configuration is shown in Fig. 7.
If coincidences or two-photon events are measured as a function of the time of arrival difference at the beam-splitter a pronounced dip is observed, i.e. the two photons never leave on either side, but always together. This dip is called the **Hong-Ou-Mandel dip** or HOM dip. Its depth (ideally down to zero) is a measure for the indistinguishability of two photons.

Figure 8: Experimental results of the Hong-Ou-Mandel experiment, the HOM-dip [Hong, Ou, and Mandel, Phys. Rev. Lett. 59, 2044 (1987)].
2.2.2 The KLM-Gate

The HOM interference together with single photon detection can be used to implement two-photon gates. The main idea of the gate proposed by E. Knill, R. Laflamme, and G. J. Milburn (Nature Vol 409, 46 (2001)), the KLM-gate, uses so-called ancilla states, which are indistinguishable single photons in additional modes (additional to the qubit modes). After the photons have passed a certain gate made from various optical elements the ancilla states are measured. The operation of the gate is accepted only if a specific outcome is measured. Otherwise the operation has to be started again. Knill et al. proposed a gate $NS_{-1}$ that performs a non-deterministic phase shift on an initial state $|\psi\rangle$ as follows:

$$|\psi\rangle = \alpha_0 |0\rangle + \beta_1 |1\rangle + \gamma_1 |2\rangle = (\alpha_0 + \beta_1 a^{\dagger} + \gamma_1 a^{(2)}\dagger) |0\rangle \rightarrow \alpha_0 |0\rangle + \beta_1 |1\rangle - \gamma_1 |2\rangle$$

The gate $NS_{-1}$ can be realized by the network of linear optical elements shown in Fig. 9:

![Figure 9: Nonlinear phase shift gate. [from Knill et al., Nature 409, 46 (2001)]](image_url)

Here the action of each of the three beamsplitters includes a phase shift explicitly, such that e.g. the transformation between the annihilation (creation) operators $a_1$ and $a_2$ before and after the beamsplitter is described by the matrix:

$$\begin{pmatrix} 
\cos \vartheta & -e^{i\phi} \sin \vartheta \\
-\sin \vartheta & e^{-i\phi} \cos \vartheta 
\end{pmatrix}$$
It can be shown that the effect of the whole network for a particular setting of \( \vartheta \), and \( \phi (\vartheta_1 = \pi/8, \phi_1 = 0, \vartheta_2 = 3\pi/8, \phi_2 = 0, \vartheta_3 = -\pi/8, \phi_3 = 0, \phi_4 = \pi) \) can be described by the following matrix, which transforms the creation (annihilation) operators in modes 1, 2, and 3:

\[
\begin{pmatrix}
1 - 2^{1/2} & 2^{-1/4} & (3/2^{1/2} - 2)^{1/2} \\
2^{-1/4} & 1/2 & 1/2 - 1/2^{1/2} \\
(3/2^{1/2} - 2)^{1/2} & 1/2 - 1/2^{1/2} & 2^{1/2} - 1/2
\end{pmatrix}
\]

At the beginning the ancilla states in mode 2 and 3 are set to:

\[ |\psi\rangle_2 = |1\rangle \]
\[ |\psi\rangle_3 = |0\rangle \]

The operation is accepted only if the detectors (at \( R_2 \) and \( R_3 \)) measure the ancilla states unchanged, i.e. exactly one photon in mode 2 and no photon in mode 3. Otherwise the operation has to start again. The probability of success is 1/4.

The \( NS_{-1} \) gate can then be used to construct a conditional phase shift as illustrated in the following picture:

![Figure 10: Conditional sign flip gate implemented with NS-1. [from Knill et al., Nature 409, 46 (2001)]](image-url)
The two beamsplitters set at $45^\circ = \frac{\pi}{4}$ are 50/50 beamsplitters. If exactly one photon is in mode 1 and in mode 3 then after the first beamsplitter the state is:

$$|\psi\rangle = |20\rangle_{13} + |02\rangle_{13}$$

Only then a phase shift of -1 occurs which is transferred to qubit $Q_2$ after the second beam splitter which completes the operation. Thus, the probability of success for the controlled phase shift is $1/16$.

One drawback of the proposal is this finite probability. However, in the same paper the authors suggest methods to boost up the success probability arbitrarily close to 1 with the help of even more ancilla states and quantum teleportation. What remains is the requirement for single photon production and for detectors which can discriminate between zero, one, and more than one photon. Additionally, the number of optical elements increases dramatically.

### 2.2.3 Another intuitive two-photon gate

The KLM proposal motivated many other suggestions for linear optical quantum computing gates. The goal is to reduce the additional effort in terms of the number of ancilla photons and optical elements.

Figure 11 shows another intuitive $NS_{-1}$-gate proposed by Rudolph and Pan using polarizing beam-splitters (PBS): Assume the input state is horizontally polarized:

$$\left( \alpha + \beta \hat{a}^+_H + \frac{\gamma}{\sqrt{2}} \hat{a}^{12}_H \right) \hat{b}^+_V |0\rangle$$

After the first polarizer the state is:

$$\left[ \alpha + \beta \cos \sigma \hat{a}^+_H + \beta \sin \sigma \hat{a}^+_V + \frac{\gamma}{\sqrt{2}} (\cos^2 \sigma \hat{a}^{12}_H + \sin 2\sigma \hat{a}^+_H \hat{a}^+_V + \sin^2 \sigma \hat{a}^{12}_V) \right] \hat{b}^+_V |0\rangle$$

Detecting no photon in the first arm yields:

$$\left( \alpha + \beta \cos \sigma \hat{a}^+_H + \frac{\gamma}{\sqrt{2}} \cos^2 \sigma \hat{a}^{12}_H \right) \hat{a}^+_V |0\rangle$$

After the second polarizer $\theta$:

$$\left[ \alpha + \beta \cos \sigma (\cos \theta \hat{a}^+_H + \sin \theta \hat{a}^+_V) + \frac{\gamma}{\sqrt{2}} \cos^2 \sigma (\cos \theta \hat{a}^+_H + \sin \theta \hat{a}^+_V)^2 \right] \times (- \sin \theta \hat{a}^+_H + \cos \theta \hat{a}^+_V) |0\rangle$$

Finally after detecting a single (!) vertically polarized photon in the second detector:

$$|\psi_{out}\rangle = \alpha \cos \theta |0\rangle + \beta \cos \sigma \cos 2\theta |1\rangle + \gamma \cos^2 \sigma \cos \theta (1 - \sin^2 3\theta) |2\rangle$$
With $\sigma \simeq 150.5^\circ$ and $\theta \simeq 61.5^\circ$ this yields the $NS_{-1}$ gate with probability

$$P_{\text{success}} = \cos^2 \theta = \frac{3 - \sqrt{2}}{7}$$

### 2.2.4 Experimental demonstrations

Several fundamental gates have been demonstrated using LOQC:
- Franson et al. Fortschr. Phys. 51, 369 (2003), Fidelity: 86%
- O'Brien et al., Nature 426, 264 (2003), Fidelity: 84%
- Pittman et al., Phys. Rev. A 68, 032316 (2003), Fidelity: 79%

Usually the gates are tested by measuring the truth table. However, the truth
table in a certain basis is a classical operation. In order to prove the coherence of the gate one has to use coherent superposition of basis states as input. Then, the output state is typically an entangled state. This is shown in the following experimentally determined density matrix of the two photon output state of the gate by O’Brian. For a complete characterization the complete map of all input states to all output states has to be determined.

![Figure 13: Measured real part of the density matrix of the gate by O’Brien et al. when the input state was a coherent superpositions of the two basis states](image)

### 2.2.5 Scalability of LOQC

The question may arise if quantum computation with probabilistic gates may be useful at all. When the gates in a computational circuit succeed only with a certain probability $p$, then the entire calculation that uses $N$ such gates succeeds with probability $p^N$. The resources time or circuits scale exponentially with the number of gates. In order to do useful quantum computing with probabilistic gates, one has to take the probabilistic elements out of the running calculation. In 1999, Gottesman and Chuang proposed a trick that removes the probabilistic gate from the quantum circuit and places it in the resources that can be prepared offline. It is commonly referred to as the teleportation trick, since it teleports the gate into the quantum circuit.

Suppose a probabilistic CZ gate needs to be applied to two qubits with quantum states $|\phi_1\rangle$ and $|\phi_2\rangle$, respectively. If the gate is applied directly to the qubits, very likely the qubits are destroyed. However, suppose that both qubits are
teleported from their initial mode to a different mode. For one qubit, this is shown in Fig. 14. Here x and z are binary variables, denoting the outcome of the Bell measurement, which determine the unitary transformation that needs to be applied to the output mode. If x=1, the x Pauli operator denoted by X needs to be applied, and if z=1, the z Pauli operator needs to be applied. If x,z=0, no operator has to be applied. For teleportation to work, the entangled resource $|\Phi^+\rangle$ is required, which can be prepared off-line. If a suitable storage device is available, $|\Phi^+\rangle$ does not have to be made on demand: it can be created with a probabilistic protocol using several trials and stored in the storage device.

![Figure 14: The teleportation circuit. The state $|\phi_j\rangle$ is teleported via a Bell state $|\Phi^+\rangle$ and a Bell measurement B. The binary variables x and z parametrize the outcome of the Bell measurement and determine which Pauli operator is applied to the output mode.](image)

One can now apply the probabilistic CZ gate to the output of two teleportation circuits. Now, the CZ gate can be commuted through the Pauli operators X and Z at the cost of more Pauli operators. That means it can be moved from the right to the left at the cost of only the optically available single-qubit Pauli gates. Again, the required resource can be prepared off-line with a probabilistic protocol and stored in a suitable storage device. There are now no longer any probabilistic elements in the computational circuit.

It is important to note that also teleportation can be implemented with linear optical elements and photon counters, as shown in the KLM proposal. Thus, using these tricks together with efficient error correction LOQC is in principle possible.

In conclusion, "...the physical resources for the original KLM protocol, albeit scalable, are daunting. For linear optical quantum computing to become a viable technology, we need more efficient quantum gates." [Kok et al., 2007]
Figure 15: The CZ gate using teleportation: here $|\psi\rangle = U_{CZ} |\phi_1 \phi_2\rangle$. By commuting the CZ gate through the Pauli gates from the computational circuit to the teleportation resources, the probabilistic part is taken off-line. The teleportation channel (the shaded area, including the CZ) can be prepared in many trials, without disrupting the quantum computation.

2.2.6 Summary: Optical photon quantum computer

- **Qubit representation**: Location of single photons between two modes $|01\rangle$ and $|10\rangle$, or polarization.

- **Unitary evolution (Kerr media)**: Arbitrary transforms are constructed from phase shifters, beam splitters, and non-linear Kerr media. The latter should establish a cross phase modulation of $\pi$.

- **Unitary evolution (LOQC)**: Single qubit gates are constructed from phase shifters, polarizers, and beam splitters. Two-qubit gates can be constructed with linear optical elements and single photon detectors. Teleportation and efficient error corrections can be implemented to bring the success probability close to unity.

- **Initial state preparation**: Create single photon states. This can be approximated by using attenuated laser light pulses, but for real optical quantum computers single photon states and ancilla states are required.

- **Readout**: Detect single photons, e.g., by an avalanche photo diode (APD).

- **Drawbacks**: Non-linear Kerr media with large ratio of cross phase modulation strength to absorption are difficult to realize. LOQC requires huge resources in terms of perfect ancilla states. Also storage media for entangled states are required.
2.3 Cavity Quantum electrodynamics

2.3.1 Jaynes-Cummings-Hamiltonian

In cavity quantum electrodynamics (cavity-QED) one attacks the dilemma of optical quantum computation: On one hand photons seem to be ideal candidates for carrying quantum information. Qubits are relatively easy encoded, and photons only interact weakly with their environment. The latter point however is the main obstacle in order to realize multiple quantum gates. In cavity-QED it is possible to obtain a strong interaction between single atoms and single photons without the problems of scattering, photon loss, or fast decoherence. Single atoms can mediate interactions between two photons and thus cause an effective strong photon-photon interaction.

An important number in cavity-QED is the Q-factor. It is the ratio of the cavity’s resonance frequency and its linewidth. The latter is inversely proportional to the photon loss rate. Q-factors exceeding $10^{10}$ have been realized with photon storage times of seconds. (Compared to acoustic frequencies, say 400 Hz, this would correspond to undamped oscillation of several months!).

The high-Q leads to several important properties:

- The modes have very narrow linewidth. Interaction with single modes can be achieved.
- The photon loss rate is very small and thus decoherence is very small.
- The electromagnetic field is enhanced on resonance. In other words photons are reflected many times from the cavity mirrors or cavity walls. Thus they have the chance to interact many times with atoms in the cavity.
- The latter holds also for the vacuum field which could lead to a strong modification of the spontaneous emission inside the cavity, particularly for cavities with a small mode volume and thus a large electric field per photon

A monochromatic single-mode electric field is quantized as follows:

$$\vec{E}(r) = i \vec{\varepsilon} E_0 \left[ a e^{ikr} - a^\dagger e^{-ikr} \right]$$

where $E_0$ is the amplitude, $\vec{\varepsilon}$ the polarization, $k = \omega/c$ is the spatial frequency, and $a, a^\dagger$ are the annihilation and creation operators. The free evolution of the field is given by the Hamiltonian:

$$H_{field} = \hbar \omega a^\dagger a$$
The free Hamiltonian of a two level atom can be written as:

$$ H_{\text{atom}} = \frac{\hbar \omega_0}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar \omega_0}{2} \sigma_3 $$

with the Pauli matrix $\sigma_3$. The atomic state lives in a two-dimensional Hilbert space and can either be in the lower state with energy $E_L$ or in the upper state with energy $E_U$. ($\omega_0 = E_U - E_L$).

The interaction between atom and field is described as a dipole interaction $\mathbf{d} \cdot \mathbf{E}$. Its quantized version in the so-called rotating wave approximation is:

$$ H_{\text{int}} = g (a^\dagger \sigma_- - a \sigma_+) $$

where $g$ is the atom-field coupling constant and $\sigma_-, \sigma_+$ are the Pauli raising and lowering operators:

$$ \sigma_+ = \sigma_1 + i \sigma_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \sigma_- = \frac{\sigma_1 - i \sigma_2}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} $$

The Jaynes-Cummings-Hamiltonian is the sum of the three Hamiltonians above:

$$ H_{JC} = H_{\text{field}} + H_{\text{atom}} + H_{\text{int}} = \hbar \omega a^\dagger a + \frac{\hbar \omega_0}{2} \sigma_3 + g (a^\dagger \sigma_- - a \sigma_+) $$

This Hamiltonian couples the two atomic states $|\text{up}\rangle = |1\rangle$, $|\text{down}\rangle = |0\rangle$ with only two photon number states $|n\rangle$ and $|n+1\rangle$. In case of only one excitation (photon) the basis states are: $|0\rangle_{\text{photon}} |0\rangle_{\text{atom}}$, $|0\rangle_{\text{photon}} |1\rangle_{\text{atom}}$, and $|1\rangle_{\text{photon}} |0\rangle_{\text{atom}}$. In this case the Jaynes-Cummings-Hamiltonian is a 3x3 matrix:

$$ H_{JC} = - \begin{pmatrix} \delta & 0 & 0 \\ 0 & \delta & g \\ 0 & g & -\delta \end{pmatrix} $$

where $\delta = (\omega_0 - \omega)/2$ is the detuning and $\hbar = 1$.

It is straightforward to show that the time evolution operator $U_{JC}$ can be written as:

$$ U_{JC} = e^{-i H_{JC} t} |00\rangle \langle 00| + \left( \cos \Omega t + i \frac{\delta}{\Omega} \sin \Omega t \right) |01\rangle \langle 01| + \left( \cos \Omega t - i \frac{\delta}{\Omega} \sin \Omega t \right) |10\rangle \langle 10| - i \frac{g}{\Omega} \sin \Omega t \left( |01\rangle \langle 10| + |10\rangle \langle 01| \right) $$
where $\Omega_R = 2\Omega = \sqrt{\delta^2 + g^2}$ is the Rabi frequency. The dynamics of a two level atom interacting with an electromagnetic field can be represented by a so-called Bloch vector pointing on a Bloch sphere. Here $|0\rangle$ could be the atom in the upper state, and $|1\rangle$ atom in the lower state. With the Bloch vector in the equatorial plane the state is a coherent superposition of the form $1/\sqrt{2}(|0\rangle + e^{i\phi}|1\rangle)$.

![Figure 16: Representation of a qubit state as a vector on a Bloch sphere](image)

### 2.3.2 Resonant Interaction

As one example the probability that a single photon is absorbed by the atom (initially in the ground state $|0\rangle$) is:

$$p_{\text{absorb}} = \sum_k |\langle 0| U_{JC} |k\rangle|^2 = \frac{g^2 \sin^2 \Omega t}{\delta^2 + g^2/2} \frac{1}{2} (1 - \cos(2\Omega t))$$

Accordingly, the probability to find the atom in the upper state oscillates. This phenomenon is called Rabi oscillation. This oscillatory energy exchange between the atom and the field mode is a quantum analog to the classical coupled pendulum problem. It replaces the monotonic and irreversible process of spontaneous emission when an atom interacts with a continuum of modes (e.g. in
free space, see figure below).

![Figure 17: Rabi flopping versus spontaneous decay](image)

**Experimental realizations:**

First experiments that aimed at the realization of a physical system which could be described by the Jaynes-Cummings Hamiltonian were performed with Rydberg atoms and superconducting resonators. Ryberg atoms are alkali atoms with a single electron in a highly excited state (n = 50 to 70). The atoms are thus similar to very large Hydrogen atoms. Since the dipole moment scales with the atoms diameter ($d = r \cdot e$) the interaction of Rydberg atoms with the electromagnetic field is very strong. At the same time their lifetime is very long as well. A transition between two neighboring Rydberg states is a very good approximation for a two level system. These transitions are in the microwave regime (some 10 GHz). Very high-Q superconducting cavities are available for these frequencies.

In the experiments performed in two groups (MPI Garching, and ENS Paris) diluted atomic beams cross different types of cavities (Fabry-Perot or cylinder cavities). The interaction time of the atoms with the field is the time of flight through the cavity. On the average there is at most one atom interacting with the cavity.
Figure 18: Schematics of the experimental setup in the Garching group (single atom maser). A superconduting microwave cavity is mounted inside a cryostat (also to reduce the number of thermal photons). An atomic oven and laser excitation produces Rydberg states. [from http://www.mpq.mpg.de/micromaser.html]

Single microwave photons cannot be detected. The output of the experiment is encoded in the atoms which have passed the cavity. Atoms which have left the cavity in the upper or lower state can be distinguished by subsequent ionization in an electric field. The detection efficiency is better than 40%.
In the Paris group the Rydberg atoms could be prepared in a coherent superposition of the upper and lower states with the help of a (classical) microwave pulse before entering the cavity. Also, behind the cavity a second microwave pulse acts as an analyzer. In this experiment a rapid tuning (on resonance/off resonance) was possible by applying an electric field between the two cavity mirrors. The resulting Stark effect is strong enough to considerably detune the atom with respect to the cavity. The figures 20 and 21 show experimental results from the ENS and the Garching group. Both experiments detected Rydberg atoms that had passed an on-resonance cavity. The probability to measure the atoms in the upper or lower state was measured as a function of the time-of-flight through the cavity. In the first experiment a weak coherent state was coupled into the cavity. In the second experiment Fock states were prepared via detection of (one, two, ...) successive Rydberg atoms in the lower state. Then, subsequent atoms detected the according Rabi oscillations.

Figure 19: Schematics of the experimental setup in the ENS group in Paris. [from Bouwmeester et al.]
Another experiment was performed in the ENS group in order to exchange an arbitrary qubit state between two successive atoms. The following sequence was applied

1. Prepare atom 1 in an arbitrary state:
   \[ |\psi\rangle = (\alpha |0\rangle_{\text{atom}} + \beta |1\rangle_{\text{atom}}) |0\rangle_{\text{field}} \]

2. Send atom 1 through the cavity with the cavity in resonance. The interaction time is set such that \( \Omega_R t = \pi \), (\( \pi \)-pulse), i.e. an atom in the excited state emits a photon with probability one:
   \[ \rightarrow |\psi\rangle = |0\rangle_{\text{atom}} (\alpha |0\rangle_{\text{field}} + \beta |1\rangle_{\text{field}}) \]

3. Prepare atom 2 in the ground state:
   \[ |\psi\rangle' = |0\rangle_{\text{atom}} (\alpha |0\rangle_{\text{field}} + \beta |1\rangle_{\text{field}}) \]
4. Send atom 2 through the cavity with the cavity in resonance. Again the interaction time is set such that $\Omega_R t = \pi$, (π-pulse), i.e. the atom in the ground state absorbs a photon with probability one:

$$\rightarrow |\psi\rangle' = (\alpha |0\rangle_{\text{atom}} + \beta |1\rangle_{\text{atom}}) |0\rangle_{\text{field}}$$

### 2.3.3 Off Resonant Interaction/Phase Shifts

The phase shift experienced by a single photon due to the presence of a single atom in the ground state can be easily derived from the time evolution operator $U_{JC}$:

$$\chi_{\text{photon}} = \text{arg} \left[ e^{-i\delta t} \left( \cos \Omega t - i \frac{\delta}{2\Omega} \sin \Omega t \right) \right]$$

For large detuning the absorption of a photon becomes very unlikely, whereas there is still a considerable phase shift.
Similarly the phase shift experienced by an atom in the ground state due to the presence of one photon is:

$$\chi_{\text{atom}} = \arg \left[ \cos \Omega t - i \frac{\delta}{\Omega} \sin \Omega t \right]$$

The phase shift of the atomic ground state due to the presence of a single photon can be used to realize a *conditional phase shift gate*. Such a gate can be used to construct a universal two qubit logic gate.

![Figure 22: Schematics of the levels used for experimental realization of a conditional phase gate](image)

Control and target qubit are encoded in two successive atoms. The experimental procedure goes as follow:

1. Prepare atom 1 (control) in a coherent superposition:

   $$|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle_{\text{atom}} + |1\rangle_{\text{atom}}) |0\rangle_{\text{field}}$$

2. Send atom 1 through the cavity with the cavity *in resonance* with respect to the $0 \longrightarrow 1$ transition (see Fig. 22), i.e. swap the atomic state to the photon state:

   $$\longrightarrow |\psi\rangle = |0\rangle_{\text{atom}} \frac{1}{\sqrt{2}} (|0\rangle_{\text{field}} + |1\rangle_{\text{field}})$$

3. Prepare atom 2 (target) in a coherent superposition:

   $$|\psi\rangle' = \frac{1}{\sqrt{2}} (|0\rangle_{\text{atom}} + |1\rangle_{\text{atom}}) \frac{1}{\sqrt{2}} (|0\rangle_{\text{field}} + |1\rangle_{\text{field}})$$

4. Send atom 2 through the cavity with the cavity *off resonance* or in resonance with a $2\pi-$pulse condition with respect to the $1 \longrightarrow i$ transition
(see Fig. 22):

\[ |\psi\rangle' = \frac{1}{2} (|0\rangle_{\text{atom}} + |1\rangle_{\text{atom}}) |0\rangle_{\text{field}} + \frac{1}{2} (|0\rangle_{\text{atom}} + e^{i\chi} |1\rangle_{\text{atom}}) |1\rangle_{\text{field}} \]

\[ = \frac{1}{2} (|00\rangle + |10\rangle + |01\rangle + e^{i\chi} |11\rangle) \]

The shift $\chi$ depends on the detuning of the $1 \rightarrow i$ transition. In case of zero detuning a photon can be absorbed by the $1$ state. In this case the interaction time is set such that $\Omega_R t = 2\pi$, (2$\pi$-pulse), i.e. the atom absorbs a photon but emits it again with probability one. Thus the state of the $1$ state of atom remains unchanged and only experiences a phase shift of $\chi = -1$.

The following shows experimental results of a conditioned-phase gate achieved in the ENS experiment:

Figure 23: Experimental results of a conditioned-phase gate. The probability to detect the atom in the ground state is measured for various cavity detunings. Open diamonds correspond to an empty cavity, solid squares are single photon fringes. [from Rauschenbeutel et al., Phys. Rev. Lett. 83, 5166]
2.3.4 Flying Qubits

A single atom can mediate the interaction between two photons very effectively. It can be regarded as an almost ideal Kerr medium to achieve cross-phase modulation. The most fundamental system to study the Kerr effect consists of a single three level atom interacting with two modes of the electromagnetic field:

![Diagram of a three level system as a Kerr medium](image)

Figure 24: Model of a three level system as a Kerr medium

For such a system (with degenerate energy levels 1 and 2) the generalized Jaynes-Cummings Hamiltonian is:

\[
H_{JC} = \sum_{i=1,2} \hbar \omega a_i^\dagger a_i + \frac{\hbar \omega_0}{2} \sigma_3 + \sum_{i=a,b} g_i (a_i^\dagger \sigma_i^- - a_i \sigma_i^+) 
\]

It can be shown that a single photon in mode \(a\) or mode \(b\) experiences a phase shift \(\phi_a\) and \(\phi_b\) if an atom in the ground state is present. This phase shift could be understood classically by the index of refraction of the atom in the cavity. However, there is an additional phase shift \(\Delta\) if both photons are present. The non-linear phase shift is plotted in the following figure depending on the atom-cavity detuning. This additional phase shift is the Kerr cross phase modulation.
An experiment where a conditional phase shift gate with the help of the Kerr effect was realized was performed in the group of Kimble at Caltech, USA. In contrast to the experiment in the ENS group the qubits were encoded in two single photons (flying qubits) in two modes. The photonic qubit states were approximated by two weak coherent states in two orthogonal modes with slightly different frequencies.

The initial state was a linearly polarized (probe) beam and a circularly polarized (pump) beam:

\[
|\psi\rangle_{in} = |\beta^+\rangle_{pump} \frac{1}{\sqrt{2}} (|\alpha^+\rangle + |\alpha^-\rangle)_{probe}
\approx \left[ |0^+\rangle + \beta |1^+\rangle \right] \frac{1}{\sqrt{2}} \left[ |0^+\rangle + \alpha |1^+\rangle + |0^-\rangle + \alpha |1^-\rangle \right]
\]

Here the +,- denotes left or right circular polarization. The atom was a Cs atom optically pumped in the $6S_{1/2}, F = 4, m_F = 4$ state.

After the atom had interacted with the cavity the photons were left in the state:

\[
|\psi\rangle_{out} \approx |0^+\rangle \frac{1}{\sqrt{2}} \left[ |0^+\rangle + \alpha e^{i\phi_a} |1^+\rangle + |0^-\rangle + \alpha |1^-\rangle \right] + \beta e^{i\phi_b} |1^+\rangle \frac{1}{\sqrt{2}} \left[ |0^+\rangle + \alpha e^{i(\phi_a + \Delta)} |1^+\rangle + |0^-\rangle + \alpha |1^-\rangle \right]
\]
The states in square brackets are linearly polarized states which are tilted off the vertical by an angle \( \phi_a \) and \( \phi_a + \Delta \), respectively. The shift of the probe beam was detected in the experiment with the help of a balanced homodyne detector (the local oscillator discriminated the pump from the probe beam).

Figure 26: Experimental demonstration of a conditional phase gate in the Caltech experiment. Qubits were encoded in photons ("flying qubits"), a strong Kerr nonlinearity was mediated by single Cs atoms. [from Turchette et al. Phys. Rev. Lett. 75, 4710 (1995)]

In the experiment the shifts \( \phi_a = 17.5^\circ \), \( \phi_b = 12.5^\circ \), and \( \Delta = 16^\circ \) were established. This experiment thus demonstrates the possibility to realize a universal two qubit quantum gate.

More recent experiments with high-Q cavities are performed in the optical domain. In these experiments atomic traps instead of atomic beams are used. Atoms are trapped and cooled above a cavity and then dropped. This increases the interaction time. However, it is extremely difficult to cascade many cavities. First results have also been obtained with semiconductor nanostructures (quantum dots) in solid-state cavities [Yoshie et al., Nature 432, 200 (2004); Reithmaier et al., Nature 432, 197 (2004); Peter et al., Phys. Rev. Lett. 95, 067401 (2005)]. Rabi oscillations could be observed. These all-solid state systems may have the potential for up-scaling towards two, three, N (?) quantum dots and/or cavities.
Figure 27: Measured phase shift in the Caltech experiment. The mean number of atoms in the cavity was 1. The inset shows the transmission of the probe through the cavity. [from Turchette et al. Phys. Rev. Lett. 75,4710 (1995)]

2.3.5 Summary: Cavity QED implementation of quantum computers

- **Qubit representation:** Location of single photons between two modes $|01\rangle$ and $|10\rangle$, or polarization.

- **Unitary evolution:** Arbitrary transforms are constructed from phase shifters, beam splitters, and cavity QED systems.

- **Initial state preparation:** Create single photon states. This can be approximated by using attenuated laser light pulses.

- **Readout:** Detect single photons, e.g. by an avalanche photo diode (APD).

- **Drawbacks:** The coupling of two photons is mediated by an atom. Thus, strong atom field coupling is desirable. Coupling of photons into and out of the cavity becomes difficult. The scalability is limited.
3 Ion Trap Implementations

Single atoms or ions in vacuum chambers represent a very well isolated and controllable quantum system. Lasers offer an almost ideal tool to manipulate and also to detect qubits which are encoded in atomic states. We will concentrate here on experiments with trapped ions. Other proposals suggest the use of neutral atoms and optical lattices.

3.1 Trapping Ions

3.1.1 Paul traps

It is well known from the so-called Earnshaw theorem that a charged particle cannot be trapped in a stable configuration with static electric fields only. However, a combination of static electric and magnetic fields (Penning trap) or time dependent electric fields (Paul trap) can provide space points where a restoring force in all three directions acts on a charged particle.

A Paul trap (Nobel Prize 1989) consists of two parabolical electrodes and a ring electrode.

Figure 28: Sketch of a Paul trap
If a dc-voltage $U_{dc}$ and an ac-voltage $V_{ac}$ of frequency $\Omega$ is applied to the electrodes then the potential near the trap axis is of the form:

$$\Phi = \frac{(U_{dc} + V_{ac}\cos(\Omega t))(r^2 - 2z^2 + 2z_0^2)}{r_0^2 + 2z_0^2}$$

where $r_0$ and $z_0$ denote the distances from the trap axis to the surface of the electrodes.

The potential is harmonic and for a certain time $t$ provides a restoring force in one dimension.

Say, at a given time $t$ the x-direction is the non-confining direction. Then, due to its inertia a particle cannot escape along that direction before the sign of $\cos(\Omega t)$ is inverted, and the x-direction becomes the confining direction. For
certain frequencies $\Omega$ this results in an effective confinement in all three directions.

The equations of motion in a Paul trap are:

$$m \frac{d^2}{dt^2} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{2q}{r_0^2} \left( U_{dc} - V_{ac} \cos(\omega t) \right) \left(\begin{array}{c} -x \\ -y \\ 2z \end{array}\right)$$

With the transformation

$$\tau = \frac{1}{2\omega},$$
$$a_x = a_y = -a_z/2 = \frac{4qU_{dc}}{mr_0^2\Omega^2},$$
$$q_x = q_y = -q_z/2 = \frac{2qV_{ac}}{m\Omega^2r_0^2},$$

one finds the Mathieu equations:

$$\frac{d^2 u_i}{d\tau^2} + (a_i + 2q_i \cos(2\tau)) u_i = 0$$

with $i = x, y, z$

For the approximation $a_i < q_i << 1$, which is usually fulfilled, there exists an
analytic solution:

\[ u_i(t) = A_i \cos(\Omega_i t + \phi_i)[1 + \frac{q_i}{2} \cos(\omega t)] \]

The solution consists of a rapid oscillation with the trap frequency \( \omega \), the so-called \textit{micromotion}, and a slower \textit{macromotion} (or \textit{secular motion}) at frequency \( \Omega_i \) in the effective harmonic trap potential with:

\[ \Omega_i \approx \frac{\omega}{2} (a_i + \frac{q_i}{2}) \]

In the trap center the micromotion vanishes completely.

As an example of a very fundamental experiment with single trapped ions Fig. 32 shows the fluorescence from a single Ba-Ion.

![Figure 31: Energy levels of Ba](image)

If a single ion is trapped and continuously illuminated with laser light (here on the \( P_{1/2} \rightarrow S_{1/2} \) transition, see Fig. refBalevels) then the fluorescence vanishes abruptly due to transitions to the metastable \( D_{5/2} \) level (The other arrow indicates repumping from the \( D_{3/2} \) level). First experiments were performed in 1986. The possibility to observe these “Quantum Jumps” led to many theoretical discussions in the early 50’s.

### 3.1.2 Trapping ion strings

In order to realize quantum computers it is required to store many ions. \textit{Linear ion traps} are modifications of the single-ion Paul trap. These traps usually consist of two pairs of electrodes (hyperbolically shaped, spherical rods, or rectangular shapes) which provide a confinement in two directions. These configurations resemble mass spectrometers (more precisely \( e/m \) filters). A second
pair of end cap electrodes provides the confinement in the orthogonal direction. Figure 33 shows various configurations.

In a linear trap the equation of motion in z-direction (direction of the end caps) is given by:

$$\frac{d^2 u_z}{dt^2} + \left( \frac{2qU_{cap}}{mz_0^2} \right) u_z = 0$$

where $q, m$ are the charge and mass of the ion and $\kappa$ is an empirically found parameter.

The equation of motion in the orthogonal directions is given again by the Mathieu equations:

$$\frac{d^2 u_i}{d\tau^2} + \left( a_i + 2q_i \cos(2\tau) \right) u_i = 0$$

with $i = x, y$
where

\[ a_x = \frac{4q}{m\Omega^2} \left( \frac{U_{dc}}{r_0^2} - \frac{\kappa U_{cap}}{z_0^2} \right) \]

\[ a_y = -\frac{4q}{m\Omega^2} \left( \frac{U_{dc}}{r_0^2} + \frac{\kappa U_{cap}}{z_0^2} \right) \]

\[ q_x = -q_y = \frac{2qV_{ac}}{m\Omega^2 r_0^2} \]

\[ \tau = \frac{\Omega t}{2} \]

Similar as in the single-ion trap the motion of the ions can be approximated as a combination of micro- and macromotion. What is important now is that the micromotion vanishes completely on the whole trap axis!

Figure 34 The following shows a collection of pictures (raw video data) of stored...
Ca-ion strings. The images were taken with CCD cameras:

Figure 34: Strings of ions stored in a linear Paul trap [from http://heart-c704.uibk.ac.at/]

3.1.3 Normal Modes

If the radial confinement in a linear ion trap is strong enough, ion arrangein linear pattern along the trap axis at low temperatures. The distance between the ions is determined by the equilibrium of the Coulomb repulsion and the potential providing axial confinement.

Small displacements of the ions from their equilibrium position cannot be described in terms of the motion of individual ions since the Coulomb interaction couples the charged particles. Instead, the motion of the string must be described in terms of normal modes.

- One example of a normal mode is the center-of-mass (COM) mode of the string. This corresponds to an oscillation of the whole string as if all ions were rigidly joined.
- Another example is the breathing mode. It describes an oscillation whether the ions move in opposite directions and leave the COM fixed.
- The spectrum of other higher order modes can be calculated from the trap parameters.

The following figure shows the experimental observation of the COM mode and the breathing mode of a string of 7 Ca atoms which were excited via the trap’s end cap electrodes:
Figure 35: Excitation of normal modes in a string of 7 Ca atoms [from Bouwmeester et al.]
3.2 Laser Cooling

If a string of trapped ions should be used for quantum computation it is required to cool ions down to the ground state of their normal modes. (More recent proposal have weakened this requirement, but cooling is yet desirable).

With laser cooling (Nobel prize for Chu, Cohen-Tannoudji, Phillips) temperatures which are far beyond reach of cryostats could be realized.

The definition of “temperature” in laser cooling requires some discussion. It is used to describe an atomic sample whose average kinetic energy $\langle E_k \rangle$ has been reduced by laser cooling. The label “temperature” now is written as:

$$\frac{1}{2}k_B T = \langle E_k \rangle$$

with Boltzmann’s constant $k_B$.

![Temperature scale](image-url)
3.2.1 Doppler Cooling

The idea of laser cooling is illustrated by the following figure:

Assume a moving atom is interacting with monochromatic laser which is red detuned from the resonance of an electronic transition of the atom. Then, the atom can only absorb photons from the laser light if it moves towards the laser and is thus tuned to resonance by the Doppler shift. The atom experiences a momentum change $\Delta p$ which causes a deceleration. After absorption the atom spontaneously emits the photon again. Since spontaneous emission is isotropic there is no momentum change on the average. Two laser beams from opposite directions can thus decelerate or cool the atomic motion in one direction. Accordingly, three pairs of laser beams establish a cooling in all three directions. The minimum temperature achievable in this way by Doppler cooling is the Doppler temperature:

$$k_B T_D = \frac{\hbar \Gamma}{2}$$

where $\Gamma$ is the natural linewidth of the transition.
The achievable deceleration is remarkable: An atom moving with a thermal velocity of 1000 m/sec can be stopped within a ms. This correspond roughly to $10^5$ g!

### 3.2.2 Harmonic potential

As pointed out above the trap potential for ions in a Paul trap is usually harmonic close to the trap’s center. The Hamiltonian of a particle in a one dimensional harmonic potential is:

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

with the definition of the creation and annihilation operator $a^\dagger$, $a$

$$a = \frac{1}{\sqrt{2m\hbar\omega}} (m\omega x + ip)$$

$$a^\dagger = \frac{1}{\sqrt{2m\hbar\omega}} (m\omega x - ip)$$

with the properties

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$

$$a |n\rangle = \sqrt{n} |n-1\rangle$$

one derives the Hamiltonian of the harmonic oscillator:

$$H = \hbar\omega \left( a^\dagger a + \frac{1}{2} \right)$$

The motional eigenstates of a trapped particle are thus harmonic oscillator eigenstates with equally spaced eigenvalues and sometimes similar as quantized lattice vibrations called phonons. The typical length scale $x_0$ is:

$$x_0 = \sqrt{\hbar/2m\omega}$$

### 3.2.3 Sideband Cooling

In a trap potential ions can be cooled down even further than the Doppler limit by the so-called sideband cooling. Figure 38 illustrates how this works:

The lowest picture shows the typical energy structure of an ion in a trap. It is a combination of the ion’s two internal states ($e$ and $g$) and the motional states. A laser is tuned to a transition to an excited state with a lower motional excitation. Spontaneous emission now causes a transition without change of the
motional state (on the average). Since the potential is harmonic this scheme applies to all pairs of neighboring motional states. Motional quanta are removed one-by-one in each optical cycle, and the ion ends up in the motional ground state which is then decoupled from the laser light.

In experimental realizations either a stabilized laser is used to resolve individual sidebands or a Raman transition where the energy difference between two incident lasers equals the energy of one phonon.

### 3.2.4 Choosing atoms

Although an ion trap is very deep (several eV potential well depth) and will hold almost every ion, only a few ions are suitable for quantum computation. The following requirements should be met:

- The electronic level structure should be simple to allow the realization of a closed two level system without the need of too many lasers.

- The levels used for the qubit transition should have a negligible decoherence (e.g. by spontaneous decay).

- The levels should allow for efficient laser cooling and detection (some strong transitions).
Because of these requirements the ions of choice have typically only one electron in the outer shell (hydrogenic ions). The two level system can either be provided by two hyperfine ground states or by a ground state and a long-lived metastable electronic state.

Most of the experiments have been done with $^{9}\text{Be}^+$ and $^{40}\text{Ca}^+$, but other possible ions are $^{138}\text{Ba}^+$, $^{25}\text{Mg}^+$, $^{199}\text{Hg}^+$, and $^{171}\text{Yb}^+$.

![Energy levels of Ca and Be](image)

Figure 39: Energy levels of Ca and Be [from Bouwmeester et al.]

Cooling of the ions starts with Doppler cooling. In $^{9}\text{Be}^+$, the ultraviolet $S_{1/2} \rightarrow P_{3/2}$ transition at 313 nm is used, while for the $^{40}\text{Ca}^+$ the corresponding transition is 397 nm. Frequency doubled Ti: Sapphire or dye lasers are used to generate the UV light. (A major advantage of a specific ion could be whether it is possible to use cheap lasers, e.g. diode lasers for the optical excitations!)

The Doppler cooling leads to a thermal state of motion with a temperature of about 1 mK. Then, it depends on the trap depth how many motional states $n_{\text{phonon}}$ are occupied at this temperature. The number ranges between $\langle n_{\text{phonon}} \rangle = 1.3 \left( \omega_{\text{trap}}/2\pi = 11 \text{ MHz} \right)$ and $\langle n_{\text{phonon}} \rangle = 50 \left( \omega_{\text{trap}}/2\pi = 140 \text{ kHz} \right)$. Subsequent sideband cooling cools the ions to the motional ground state.
3.3 Single and Two Qubit Gates

3.3.1 Hamiltonian of ions in a trap

The Hamiltonian of \( N \) ions in a 3-dimensional harmonic trap potential which interact via Coulomb interaction is:

\[
H_{\text{Nions}} = \sum_{i=1}^{N} \frac{m}{2} \left( \omega_{x} x_{i}^2 + \omega_{y} y_{i}^2 + \omega_{z} z_{i}^2 + \frac{|p_{i}|^2}{2} \right) + \sum_{i=1}^{N} \sum_{i<j} \frac{e^2}{4 \pi \epsilon_0 (r_{i} - r_{j})}
\]

Since a typical trap potential in a linear ion trap is shallowest in the \( z \)-direction (i.e. perpendicular to the end caps) it is sufficient here to consider only the \( z \)-coordinate. The ions remain in the ground state with respect to the oscillation in both \( x \)- and \( y \)-direction.

The ions are laser cooled to the so-called Lamb-Dicke regime. The Lamb-Dicke regime is defined as \( \eta \ll 1 \), where \( \eta \) is the Lamb-Dicke parameter. This parameter gives the ratio between the typical length scale \( z_0 \) of an ion’s oscillation amplitude in a harmonic trap potential and the wavelength \( \lambda \) of the incident laser radiation:

\[
\eta = \frac{2 \pi z_0}{\lambda} = 2 \pi \sqrt{\hbar/2Nm \omega_{\text{trap}}/\lambda}
\]

where \( N \) is the number of ions in the trap, \( m \) the ion’s mass, and \( \omega_{\text{trap}} \) is the trap frequency.

If the ions are laser cooled they only perform small oscillations around their equilibrium position. Then the Coulomb potential can be expanded as a Taylor series. Thus, the potential (trap and Coulomb potential) along the \( z \)-direction can be written as:

\[
V(z) = \sum_{i=1}^{N} \sum_{j=1}^{N} U_{ij} z_{i} z_{j}
\]

The diagonalization of this potential can be achieved by transforming to the normal variables. Finally, the Hamiltonian \( H_{\text{Nions}} \) is transformed into

\[
H_{\text{Nions}} = \sum_{i=1}^{N} \hbar \omega_{i} a_{i}^{\dagger} a_{i}
\]

which is an harmonic oscillator potential for \( N \) normal modes.

3.3.2 Interaction with a laser field

The interaction of the ions in the trap with a classical laser field at frequency \( \omega \) is now:

\[
H_{I}^{\prime} = \Omega_{i} \cos(kz_{i} + \phi_{i} + \omega t)(\sigma_{i}^{+} + \sigma_{i}^{-})
\]
Note here that we considered one particular normal mode $i$ and assumed a standing wave laser field. The Rabi-frequency $\Omega_i$ is proportional to the amplitude of the classical laser field. $\sigma_i^+$ and $\sigma_i^-$ describe the raising and lowering operators of the atom’s two internal states.

The coordinate $z_i$ is now quantized due to the motion in the trap potential:

$$z_i = z_{i,\text{equilibrium}} + \frac{1}{\sqrt{\hbar/2Nm\omega_{CM}}} (a + a^\dagger)$$

Note: We consider only an excitation of the lowest (center-of-mass) normal mode. This introduces the factor of $1/\sqrt{N}$ to the normal coordinates. With the equilibrium position absorbed in the phase $\phi_i$ the Hamiltonian reads:

$$H_i = \Omega_i \cos(\eta \sqrt{N} (a + a^\dagger) + \phi_i + \omega t) (\sigma_i^+ + \sigma_i^-)$$

In the limit of small Lamb-Dicke parameter, $\eta \ll 1$, one finds after some algebra two cases:

1. Laser tuned to the internal atomic transition $\omega = \omega_0$:

$$H_i = \Omega_i/2 \left( \sigma_i^+ \exp(i\phi_i) + \sigma_i^- \exp(-i\phi_i) \right)$$

The laser field introduces only transitions between internal states of the ions.

2. Laser tuned to one of the sideband transitions $\omega = \omega_0 \pm \omega_{CM}$:

$$H_i = \frac{\Omega_i}{2\sqrt{N}} \left( \sigma_i^+ a^\dagger \exp(i\phi_i) - \sigma_i^- a \exp(-i\phi_i) \right) \quad \text{if} \quad \omega = \omega_0 + \omega_{CM}$$

$$H_i = \frac{\Omega_i}{2\sqrt{N}} \left( \sigma_i^+ a \exp(i\phi_i) - \sigma_i^- a^\dagger \exp(-i\phi_i) \right) \quad \text{if} \quad \omega = \omega_0 - \omega_{CM}$$

In this case, in addition to an internal transition one phonon is created or annihilated.

The following figure shows a sketch of the different possibilities:
3.3.3 Single qubit operation
A qubit is encoded in the internal states of the ions:

Single qubit gates are established by tuning the laser to the frequency $\omega = \omega_0$. By choosing the phase shift $\phi$ and the duration of the interaction appropriately, arbitrary rotations can be performed. Thus, any single qubit-gate can be realized this way.
3.3.4 Two qubit operation

A controlled phase-flip gate can be constructed with the help of an auxiliary atomic level in the following way [J. I. Cirac, P. Zoller, Phys. Rev. Lett. 74, 4091 (1995)]:

1. We first assume that initially one qubit is stored in an ion's internal state ($|0\rangle$, or $|1\rangle$), another qubit is stored in the phonon state ($|0\rangle$, or $|1\rangle$). Both qubits can be in any arbitrary superposition.

2. A laser is tuned to the frequency $\omega_{aux} + \omega_z$, to cause the transition between the auxiliary state $|20\rangle$ and only the state $|11\rangle$. Because of the uniqueness of this frequency, no other transitions are excited. The phase and duration of the laser pulse is chosen properly to make a $2\pi$-pulse. This results in:

$$|11\rangle \rightarrow - |11\rangle$$

All other states remain unchanged. Thus the effect on the initial state is:

$$(|0\rangle + |1\rangle)_{\text{ion}} (|0\rangle + |1\rangle)_{\text{phonon}} = |00\rangle + |10\rangle + |01\rangle + |11\rangle$$

$$\rightarrow |00\rangle + |10\rangle + |01\rangle - |11\rangle$$

which is the desired controlled phase gate.

3. In order to decode both qubits in ions a SWAP-gate is required which maps an ion's qubit state on a phonon qubit state. This can be done by tuning the laser to the frequency $\omega_0 - \omega_z$, and arranging the phase and pulse duration such that a $\pi$-pulse is established:

$$(\alpha |0\rangle + \beta |1\rangle)_{\text{ion}} \rightarrow (\alpha |0\rangle + \beta |1\rangle)_{\text{phonon}}$$
The interaction between arbitrary qubits is achieved since the phonons are quantized modes of the center-of-mass (COM) oscillation shared by all ions in the trap! The COM-mode acts as a quantum bus as sketched in the following cartoon:

![Cartoon of quantum computation with trapped ions](image)

Figure 43: Quantum computation with trapped ions [from Bouwmeester et al.]

A CNOT gate between ion $k$ and ion $j$ can thus be constructed using the following sequence of operations:

$$CNOT_{jk} = H_k \text{SWAP}_k C_j \text{SWAP}_k H_k$$

where $C_j$ is the controlled phase gate on the ion $j$ and $H_k$ are Hadamard gates on ion $k$.

Any quantum computation can thus be performed with a string of trapped ions!

More recent proposals [J. F. Poyatos, J. I. Cirac, P. Zoller, Phys. Rev. Lett. 81, 1322 (1998)] show that clever techniques exist to perform a quantum computation with somewhat hotter ions, which don’t have to be cooled to the Lamb-Dicke regime by Doppler and sideband cooling. For these techniques Doppler cooling alone may be sufficient.

### 3.4 Experimental Realization of a CNOT Gate

A first demonstration of a CNOT gate was demonstrated in the group of D. Wineland at NIST, Boulder, USA [C. Monroe et al., Phys. Rev. Lett. 75, 4715 (1995)]. (See also http://www.bldrdoc.gov/timefreq/ion/index.htm) A single...
$^9\text{Be}^+$ ion was trapped and laser cooled into the motional ground state. The computational basis was:

$$|0\rangle|\uparrow\rangle, |0\rangle|\downarrow\rangle, |1\rangle|\uparrow\rangle, \text{ and } |1\rangle|\downarrow\rangle$$

where $|0\rangle$ and $|1\rangle$ denote the motional states and $|\uparrow\rangle$ and $|\downarrow\rangle$ the internal (hyperfine states). More precisely:

$$|\uparrow\rangle = \frac{2}{S_1}{|F = 2, m_F = 2\rangle}$$
$$|\downarrow\rangle = \frac{2}{S_1}{|F = 1, m_F = 1\rangle}$$

An additional state was used as auxiliary state,

$$|\text{aux}\rangle = \frac{2}{S_1}{|F = 2, m_F = 0\rangle}$$

another state was used for detection:

$$^2P_{3/2}|F = 3, m_F = 3\rangle$$

The following shows the level scheme and a miniaturized Be-ion trap:

![Energy levels of a Be-ion used for experimental demonstration of a CNOT gate](from Bouwmeester et al.)
In order to demonstrate a CNOT gate the following procedure was used:

1. Doppler and sideband cooling of the ion in the $|0\rangle|\downarrow\rangle$ state (95% probability).

2. Initialization in one of the four basis states $|0\rangle|\downarrow\rangle$, $|0\rangle|\uparrow\rangle$, $|1\rangle|\uparrow\rangle$, or $|1\rangle|\downarrow\rangle$ using single qubit rotations.

3. $\pi/2$-pulse on the internal state. This leaves the motional state unchanged.

4. $\pi$-pulse on the $|1\rangle|\uparrow\rangle \rightarrow |aux\rangle$ transition. All other states remain unchanged.

5. $-\pi/2$-pulse on the internal state. This leaves the motional state unchanged.
6. Detection of the population in the $|0\rangle |\downarrow\rangle$ and $|1\rangle |\downarrow\rangle$ states by shining $\sigma^+$-polarized light resonant to the $^2P_{3/2}$ transition. This measures the internal qubit.

7. Swapping the motional and internal qubit. Then repeat step 6. This measures the motional qubit.

It is easy to verify that the steps 3 to 5 implement a CNOT gate. The following shows the measured CNOT true table:

![Quantum Controlled-Not](image)

Figure 46: Experimental data of true table for a CNOT gate. [from http://www.bldrdoc.gov/timefreq/ion/index.htm]

In order to show that a CNOT gate could be performed on a coherent superposition of qubits with the coherence maintained, the detuning of the $\pi/2$-pulses was changed. The following shows the resulting so-called Ramsey-fringes for the $|0\rangle |\downarrow\rangle$ and $|1\rangle |\downarrow\rangle$ states.
The time for the CNOT operations was about 50 microseconds with a measured decoherence time of milliseconds. The main source of decoherence was due to instabilities in the laser intensity, the RF fields and magnetic fields.
3.5 Gates and Tricks with Single Ions

First quantum computing algorithms have been realized, such as the Deutsch-Jozsa algorithm in the Innsbruck group [S. Gulde et al., Nature 421, 48 (2003)]. Presently, it is possible to manipulate and individually control up to 10 ions in an ion trap. Quantum teleportation of an unknown quantum state to another was demonstrated by the Innsbruck group [M. Riebe et al., Nature 429, 734 (2004)] and Boulder group [M. D. Barret et al., Nature 429, 737 (2004)]. Also a quantum byte [Haffner et al., Nature 438, 643 (2005)] and a 6-ions GHZ entangled state [D. Leibfried et al., Nature 438, (2005)] were created. In order to obtain even more complex elements novel miniturized ion traps are being developed.

3.5.1 Demonstration of Deutsch-Jozsa

This algorithm was demonstrated by using two qubits. One qubit was implemented in the ground and a metastable state of Ca ions, the $S_{1/2}$- and $D_{5/2}$-state respectively. The following picture shows the energy diagramme.

Figure 48: Qubits in the experimental demonstration of the two-qubit Deutsch-Jozsa algorithm [from S. Gulde et al., Nature 421, 48 (2003)].

The algorithm (including the function $U_f$) was encoded by using different laser pulses. Laser frequencies could be tuned with the help of acousto-optic modulators. The following figure shows the probability to find the ion in the upper atomic qubit state. In order to derive this probability the procedure was repeated many times and interrupted after a certain time had elapsed. The experiment shows that it is indeed possible to control a single ion in an ion trap with very high precision. For more complex algorithms the application of the laser pulses will be controlled by a computer. This somewhat approaches techniques which already exist in nuclear magnetic resonance experiments for microwave pulses.
3.5.2 Teleportation of atomic qubits

A recent progress was the demonstration of teleportation of an atomic state from one trapped ion to another. In the Innsbruck group led by Rainer Blatt [M. Riebe et al., Nature 429, 734 (2004)] three ions were manipulated with carefully designed laser pulses. Individual ions could be prevented from disturbance due to detection light by applying so-called "hide"-pulses, which transferred the state to another Zeeman level. The following picture shows the diagram of the algorithm.

The implementation of the algorithm consisted of 35 laser pulses including spin echo pulses to revert dephasing. This clearly demonstrates that computer control is essential for an upscaling to more qubits. The following pictures shows
the fidelities for teleportation of different quantum states.

Figure 51: Experimental results for the teleportation of different quantum states [from M. Riebe et al., Nature 429, 734 (2004)].

The group in Boulder led by Dave Wineland followed a slightly different approach. They used a miniaturized ion trap which allowed to move ions from one segment to another. In this way it was also possible to address individual ions with laser pulses. Qubits were encoded in two hyperfine levels of Be ions.
In order to demonstrate successful teleportation of arbitrary states a Ramsey interference was measured by applying pulses to ion 1 before and to ion 3 after teleportation.
3.5.3 A Quantum Byte

Linear ion straps allow to trap and cool a large number of atoms. An experimental challenge is to manipulate all trapped ions precisely with tailored laser pulses. Recently, the Innsbruck groups realized a quantum byte, i.e., an arbitrary quantum state consisting of 8 quantum bits [Häffner et al., Nature 438, 643 (2005)].

As an example for a complex multi-particle qubit state a W-state or Werner-state was created. Such a state has a single excitation (one ion in an excited state), but there is no information in which ion.

\[ |W_N\rangle = |10000000\rangle + |01000000\rangle + \ldots |00000001\rangle \]

In the experiment the 0- and 1-state were realized as the \( S_{1/2} \) ground state and the \( D_{5/2} \) metastable state in \( ^{40}\text{Ca} \)-ions. First, all ions were prepared in the state:

\[ |W_N\rangle = |0, DDDDDDDDD\rangle \]

where the first 0 denotes the motional ground state of center of mass oscillation. Then, ion 1 is flipped by a \( \pi \)-pulse on the carrier transition to

\[ |W_N\rangle = |0, SDDDDDDDD\rangle \]

Then, most of the population is moved to the \( |W_N\rangle = |1, DDDDDDDDD\rangle \) state by applying a blue sideband pulse of length \( \theta_n = \arccos(1/\sqrt{n}) \):

\[ \frac{1}{\sqrt{N}} |0, SDDDDDDDD\rangle + \frac{\sqrt{N-1}}{\sqrt{N}} |1, DDDDDDDDD\rangle \]

This procedure is continued until the final state is reached:

\[ |W_N\rangle = \frac{1}{\sqrt{N}} |0, SDDDDDDDD\rangle + \frac{1}{\sqrt{N}} |0, DSDDDDDD\rangle + \ldots + \frac{1}{\sqrt{N}} |0, DDDDDDDS\rangle \]

The entangled procedure took about 1 ms. A major problem is to gain full information of the N-ion complex quantum state. This was obtained via quantum state reconstruction by expanding the density matrix in a basis of observables and measuring the corresponding expectation values. In order to do this, a large number (about 650000) additional laser pulses were employed. A full characterization of the state required 10 hours measurement time. A fidelity exceeding 70% were obtained (see reconstructed density matrix in Fig. 54).

The experiment demonstrates that presently there is not only a limit with respect to constructing a complex quantum state, but also to characterize is to full extend, not to mention to perform a tomography of a whole complex quantum gate.
Figure 54: Absolute values, $|\rho|$, of the reconstructed density matrix of a W-state as obtained from quantum state tomography. Ideally, the blue coloured entries all have the same height of 0.125; the yellow colored bars indicate noise. In the upper right corner a string of eight trapped ions is shown. [Häffner et al., Nature 438, 643 (2005)]
3.5.4 Novel Trap Designs

The problem of ion traps as quantum computing devices is their complexity. Ion traps have to be implemented in rather large ultra-high vacuum chambers. The required equipment for lasers and laser control is demanding as well. Figure 55 shows part of the experimental setup (mainly the vacuum chamber), and Fig. 56 a close-up of a linear ion trap.

Figure 55: Left: Part of the setup of the ion trap experiment in Innsbruck. Right: Look inside the ultra-high vacuum chamber [from http://heart-c704.uibk.ac.at/].

Figure 56: Close-up of a linear ion trap [from http://heart-c704.uibk.ac.at/].
A solution to reduce the complexity of ion-trap experiment is to develop miniaturized ion-traps, or *micro-traps* (similar approaches are pursued with traps for neutral atoms as well). Figure 57 shows a standard linear ion trap with four rods together with a layered design suitable for miniaturization. An even simpler design is displayed in Fig. 58.

Figure 57: a: Standard four-rod ion trap; b: radial pseudopotential contours; c: Implementation of four-rod design using a layered structure [Chiaverini et al., Quant. Inf. and Comp. 5, 419 (2005)].
Minaturized surface traps are combined with miniaturized waveguides/antenna for RF radiation required for Paul traps and integrated on chips using standard procedures. Figure 59 shows such a atom chip.

Figure 58: Schematic diagram of a modified two-layer cross to maintain a trapping potential at the center of the cross [Chiaverini et al., Quant. Inf. and Comp. 5, 419 (2005)].

Figure 59: Micrograph of a five wire, one-zone linear trap fabricated of gold on fused silica. The top figure is an overview of the trap chip showing contact pads, onboard passive filter elements, leads, and trapping region. The lower image shows a detail of the trapping region [Chiaverini et al., Quant. Inf. and Comp. 5, 419 (2005)].
It is indeed possible to trap ions on atom chips and to manipulate them in a controlled way (see Fig. 60). A problem is that ions couple strongly to charges and charge fluctuations. Thus, close to surfaces there is an increased heating and decoherence rate which makes it more difficult to maintain coherent operations.

Figure 60: Images of 1, 2, 3, 6, and 12 ions confined in a surface-electrode trap. The length scale is determined from a separate image of the electrodes whose dimensions are known. The horizontal bars indicate the separation distance between the ions as predicted from the measured axial oscillation frequency. The ratio between transverse and axial oscillation frequencies makes it energetically favorable for the 12 ion string to break into a zigzag shape [S. Seidlin et al., PRL 253003 (2006)]
Figure 61 displays another design for a minaturized ion trap. It can be implemented in a standard chip holder to connect all required electrodes (see Fig. 62).

Figure 61: Trap geometry of a minaturized linear ion trap [courtesy F. Schmidt-Kaler].

Figure 62: Components for assembling a minaturized ion trap in a chip holder. The inset shows a photo of five trapped ions [courtesy F. Schmidt-Kaler].
There are theoretical proposals how efficient quantum computing can be performed with a large number of ions in minaturized ion traps using only a few ions at a time [Kielpinsky et al., Nature 417, 709 (2002)]. This requires transferring ions to specific locations on an atom chip.

Figure 63: Schematic diagram of a proposed large-scale ion trap array for quantum information processing. Segmented control electrodes allow logic ions (lighter-colored circles) to be transferred to memory and processor regions. After transport ions are recooled by sympathetic cooling (darker-colored circles). [Chiaverini et al., Quant. Inf. and Comp. 5, 419 (2005)]
3.6 Ion trap Cavity QED Systems

One of the problems in the cavity QED-systems introduced in the previous chapters was that the qubits were flying qubits (atomic beams were used!). In order to solve this problem there are now attempts to use stored ions in combination with optical cavities.

Two groups in Munich and Innsbruck have successfully stored ions in an ion trap inside high-Q optical cavities.

The Innsbruck group experimentally examines the interaction of a single trapped Calcium ion with a single mode of radiation in a high finesse cavity resonant with the $S_{1/2} - D_{5/2}$ quadrupole transition of the ion. The following pictures show the experimental setup.

![Experimental setup of the ion trap cavity-QED experiment in Innsbruck](http://heart-c704.uibk.ac.at/)

Figure 64: Experimental setup of the ion trap cavity-QED experiment in Innsbruck [from http://heart-c704.uibk.ac.at/]
Figure 65: Photo of a Paul trap inside an optical resonator [from http://heart-c704uibk.ac.at/]
The Munich group have successfully demonstrated that they can place a single trapped Calcium ion at will inside an optical resonators [G. Guthohrlein et al., Nature, 414, 49 (2001)]. The following pictures show a sketch of the experimental setup and a measurement of fluorescence from the ion. Depending on the position of the ion inside the cavity, the ion is in interaction with a node or an antinode of the standing wave cavity field. Thus, the amount of detected fluorescence light reflects the mode structure of the cavity.

Figure 66: Sketch of the experimental setup in the Munich experiment [from Guthörlein et al., Nature 414, 49 (2001)]
It may be very difficult to achieve interaction of many ions with a single or a few modes in such setups. However, these systems may be promising candidates for smaller scale quantum interfaces, where quantum information is stored or manipulated in ions, then transferred to photons, and transmitted to other interfaces. The following shows a sketch of one building block of such a quantum network:

Figure 68: Sketch of a possible quantum network using cavity-QED systems with trapped ions [from Bouwmeester et al.]
3.7 Summary: Ion trap implementation of quantum computers

- **Qubit representation:** Hyperfine ground or metastable states of an atom, and lowest level vibrational modes (phonons) in the trap potential.

- **Unitary evolution:** Arbitrary transforms are constructed from application of laser pulses which externally manipulate the atomic state via the Jaynes-Cummings interaction. Qubits interact via a shared phonon state.

- **Initial state preparation:** Cooling of the atoms into their motional ground state by laser cooling.

- **Readout:** Measure population of the atomic states via fluorescence from an idler state with near unity efficiency.

- **Drawbacks:** Phonon lifetimes are short, ions are difficult to prepare in their motional ground state (many ions tend to form zig-zags or more complicated shapes); some heating mechanisms are unclear. Complexity of laser control increases with number of ions, but atom/ions chips may allow further scaling.
4 Solid State Implementation of Quantum Computers

4.1 Superconducting Qubits

In the following chapter the possible realization of quantum computing devices with conducting nanostructures is outlined. A review of various proposals and techniques can be found in Y. Makhalin et al., Rev. Mod. Phys. 73, 357 (2001), in the special issue of Fortschr. Phys. 48 No. 9-11 (2000), and in Physics Today, Nov. (2005), 42-47.

4.2 Josephson Charge Qubits

4.2.1 Single qubit gates

Charge qubits are represented by different numbers of Cooper-pair charges on a small island. The superconducting state maintains a well defined phase due to energetic separation from the unpaired electrons (Cooper-pair energy gap of the BCS ground state). This gap suppresses decoherence caused by scattering with other electrons. For this reason single (unpaired) electrons can not be used as qubits.

The following figure shows the simplest design of a Josephson charge qubit.

Figure 69: Josephson charge qubit. [from Y. Makhlin et al.]

It consists of a small superconducting island with $n$ excess Cooper-pair...
charges (relative to some neutral reference state), connected by a tunnel junction with capacitance $C_J$ and a Josephson energy $E_J$ to a superconducting electrode. A control gate voltage $V_g$ is coupled to the system via a capacitance $C_g$.

**Required temperature:**
Capacitances as small as $C_J \leq 10^{-15} \, F$ can be made. With the single-electron charging energy $E_C = e^2/(C_g + C_J)$ this requires temperature below 1 $K$.

**Reminder Josephson effect:**
The Josephson effect occurs when two superconductors are separated by a thin insulator. Tunneling through the insulator is possible, thus the problem is equivalent to a general tunneling problem. The problem can be described by the following Schrödinger equations for the two superconducting states $\Phi_1$ and $\Phi_2$:

$$i\hbar \frac{\partial \Phi_1}{\partial t} = \frac{qU}{2} \Phi_1 + E_J \Phi_2$$
$$i\hbar \frac{\partial \Phi_2}{\partial t} = -\frac{qU}{2} \Phi_2 + E_J \Phi_1$$

where $U$ describes the voltage applied across the junction.

With the ansatz
$$\Phi_i = \sqrt{n_i} \exp(\phi_i) \text{ with electron density } n_i = |\Phi_i|^2$$

one finds:

$$n_1 = \frac{E_J}{\hbar} \sqrt{n_1 n_2} \sin(\phi_2 - \phi_1) = -\dot{n}_2$$
$$\hbar(\dot{\phi}_2 - \phi_1) = -qU$$

These equations describe the DC- and AC Josephson effect: Persistent current without an applied voltage, and AC-current with a DC-voltage applied.

If the energy gap is the largest energy in the system, then, at low temperatures, only Cooper-pairs tunnel coherently. The Hamiltonian of the Josephson-qubit is given by:

$$H = 4EC(\hat{n} - n_g)^2 - E_J \cos \hat{\Theta}$$
$$= \sum_n \left\{ 4EC(n - n_g)^2 |n\rangle \langle n| - \frac{1}{2} E_J (|n\rangle \langle n| + |n+1\rangle \langle n| + |n\rangle \langle n|) \right\}$$

Here, $\hat{n}$ is the number operator (of Cooper-pairs on the island), and $\Theta$ describes the phase of the superconducting state on the island. Note that $n_g = C_g V_g/2e$ is a continuous variable. The energy diagram is plotted in the following figure:
In the vicinity of \( n_g = 1/2 \) only \( n = 0 \) and \( n = 1 \) have to be considered and the Hamiltonian can be written as

\[
H = -\frac{1}{2}B_z\sigma_z - \frac{1}{2}B_x\sigma_x
\]

with

\[
B_z = 4E_C(1 - 2n_g)^2
\]
\[
B_x = E_J
\]

Obviously arbitrary unitary single qubit rotation can be performed with this Hamiltonian. Since the Josephson energy \( E_C \) is relevant only in the vicinity of \( n_g = 1/2 \) the "field" \( B_x \) can be switched on and off by changing the gate voltage \( V_g \) and thus \( n_g \).

An improved design, where \( B_z \) and \( B_x \) can be controlled independently (as required for an ideal single qubit Hamiltonian) is shown in Fig. 71: It consists of a superconducting loop with two Josephson junctions similar as in a SQUID (superconducting quantum interference device). The external flux \( \Phi_x \) which is controlled via an external magnetic field now determines the Josephson energy \( E_J \):

\[
E_J = B_x = 2E_{J0}\cos \left( \frac{\Phi_x}{\Phi_0} \right)
\]

where \( \Phi_0 = \hbar c/2e \) is the flux quantum.

Now, by changing the gate voltage \( V_g \) and the external field \( (\Phi_x) \), respectively, \( B_z \) and \( B_x \) can be controlled independently.
4.2.2 Experimental Results on Single qubit rotation:

A breakthrough experiment was performed in 1999 by Nakamura (Y. Nakamura et al., Nature 398, 786 (1999)).

The following setup was used:
The experiment realizes a SQUID-type Josephson charge qubit. Two gate electrodes are used to set the bias voltage ($V_g$ and $V_p$). The voltage $V_p$ is applied in order to quickly switch to a degeneracy point ($n_g \simeq 1/2$). Cooper-pairs on the box leaked out via the probe gate (labeled $b$) with a time constant of about 6 $n$ sec. This was detected as a current via the probe port. The experiment was performed in a dilution refrigerator at 30 $mK$ as follows:

- Initially the box was biased to a point where it was in the zero Cooper-pair eigenstate $|0\rangle$.
- A voltage pulse was switched on for a time $\Delta t$ to the degeneracy point, where mixing between the two states $|0\rangle$ and $|2\rangle$ occurred.
- Finally, after the voltage pulse the current via the probe port was detected.
- The measurement was performed as a function of the pulse length and the pulse height.

Figures 74 and 75 show the experimental results.
Figure 74: a) Energy diagram illustrating the two charge states as a function of the total gate-induced charge $Q_0 = C_g V_g + C_b V_b$. b) Schematic pulse shape. c) Probe current versus $Q_0/e$. [from Y. Nakamura et al., Nature 398, 786 (1999)]
Figure 75: a) Measured pulse-induced current as a function of $\Delta t$ and $Q_0/e$. b) numerical simulation [from Y. Nakamura et al., Nature 398, 786 (1999)]
If the difference between the probe current with and without voltage pulse is plotted as a function of pulse duration $\Delta t$ then a pronounced Rabi oscillation shows up. It has to be mentioned that the decoherence time was determined by the coupling to the probe gate!

Figure 76: Rabi oscillation in the measured pulse-induced current. The Rabi frequency is determined by the Josephson energy $E_J$ which could thus be measured as well (inset). [from Y. Nakamura et al., Nature 398, 786 (1999)]
4.2.3 Two-qubit gates

In order to perform two-qubit logic gates two qubits have to be coupled. Ideally, the coupling can be controlled. The following shows a schematic of a proposed idea.

![Schematic of coupled Josephson qubits](image.png)

Figure 77: Proposed structure for coupled Josephson qubits. [from Makhlin et al.]

Coupling between to arbitrary qubits is established by oscillator modes in an LC circuit formed by the inductor $L$ and the qubit capacitors. The Hamiltonian for the chain or register of $N$ qubits is:

$$H = \sum_{i=1}^{N} \left\{ \frac{(2en_i - C_g V_{gi})^2}{2(C_J + C_g)} - E_J(\Phi_{xi}) \cos \Theta_i \right\} + \frac{1}{2NC_{qb}} \left( q - \frac{C_{qb}}{C_J} \sum_{i} 2en_i \right)^2 + \frac{\Phi^2}{2L}$$

where $C_{qb} = C_JC_g/(C_J + C_g)$, $\Phi$ the flux through the common LC-loop, and $q$ the total charge accumulated on the gate capacitors of the array of qubits. In the case that the oscillator modes of the LC-circuit are only virtually excited ($\hbar \omega_{LC} \gg E_J, \delta E_{ch}$) the Hamiltonian can be reduced to (see Makhlin et al.):

$$H = -\sum_{i<j} \frac{E_J(\Phi_{xi})E_J(\Phi_{xj})}{E_L} \sigma_i^y \sigma_j^y$$

This Hamiltonian is of the desired form for the realization of two-qubit gates.

*Note that the two qubit interaction can be switched on and off between arbitrary qubits of the chain!*
Faster qubit operation is achieved if real oscillator quanta of the LC circuits are excited. In this case the LC-circuit acts as a quantum data bus, and the role of the excited oscillator quanta is similar to the phonons in the ion trap quantum computer.

4.3 Josephson Flux Qubits

Charge qubits are defined in a low capacitance regime, where the charging energy dominates over the Josephson energy $E_C \gg E_J$. In the opposite regime $E_C \ll E_J$ the device resembles a SQUID and the flux states are the appropriate eigenstates. The Hamiltonian can be written as

$$H = -E_J \cos \left( \frac{\Phi}{\Phi_0} \right) + \frac{(\Phi - \Phi_x)}{2L} + \frac{Q}{2C_J}$$

where $\Phi_x$ is an external flux, $L$ the self-inductance, and $Q = -i\hbar \partial / \partial t$ the charge. This Hamiltonian again can be written as

$$H = -\frac{1}{2} B_z \sigma_z - \frac{1}{2} B_x \sigma_x$$

with some $B_z(\Phi_x)$ and $B_x(E_J)$.

Qubits are now two clock- and counter-clockwise running persistent currents such that the enclosed flux is an integer multiple of $\Phi_0 = \hbar/2e!$ Single qubit rotations can be performed by switching $B_x(E_J)$ on and off. The following shows the schematics of a flux qubit and its equivalent circuit together with an energy diagram.

Figure 78: Design of a flux qubit, (a) principle, (b) with controllable Josephson energy [from Makhlin et al.]
The energy diagram reveals a double-well potential. Without going into detail the following gives a schematics of two coupled Josephson flux qubits. The dashed line induces a direct inductive coupling, whereas, alternatively, coupling is provided by an external $LC$ circuit. It can be shown that the coupling introduced in this way allows for controlled two-qubit operation, and thus for arbitrary two-qubit gates.

4.4 Readout of Josephson Qubits with SETs

An important issue is the "quiet" read-out of charge or flux qubits. So far a permanent coupling to some probe circuit has been introduced. An Ohmic and thus lossy current in the probe circuit introduces strong decoherence. In order to realize a readout that can be switched on and off it has been proposed to use
a capacitively coupled single-electron transistor: In a single-electron transistor

(SET) the large Coulomb charging energy in a small island $N$ (separated from the leads by insulating barriers with capacitance $C_T$) allows for electron tunneling only when a certain bias voltages is applied to the gate $C_g$.

The tunneling current that flows increases with the applied bias voltage and is extremely sensitive to the gate voltage. Detection of a single charge is possible. Furthermore, by changing the gate voltage, the SET can be completely decoupled from the island, and thus decoherence is reduced. Figure 82 and 83 illustrate the operation of an SET.

Figure 81: Design for a device where a charge qubit is measured with an SET [from Makhlin et al.]
Figure 82: SET current versus gate voltage ($V_g$) and bias voltage ($V_b$)

Figure 83: Working point for single charge detection with an SET
Finally, just as an impression of how Josephson quantum computing devices may look like an SET device in a standard microchip holder is shown together with a mount that can be lowered into a dilution refrigerator:

Figure 84: SET device in a standard microchip holder.

Figure 85: Lower part of the cooling finger of a dilution refrigerator.
4.5 Silicon-based Quantum Computation/Spintronics

Large scale conventional computation only became possible when thousands of transistors could be integrated onto a single solid state chip. Similarly, it is an attractive idea to implement scalable quantum computers as solid state quantum logical devices. Purest materials and most advanced technology is provided by silicon technology. Since the qubit is usually represented by the spin (nuclear or electron) the implementation of quantum computing devices in the solid state is a part of the new field of spintronics. The main focus of spintronics is the realization of devices which do not rely on the charge (electronics!) but on the spin of electrons and holes.

Here we concentrate on a proposal for solid state quantum computation using donor impurities in silicon. Details can be found in B.E. Kane, Nature, 393, 133 (1998).

The proposal by Kane suggests to use single phosphorus (P) atoms implanted as impurities in an silicon crystal (Si). The addition of Column V donors in the Si crystals results in electron states near in the energy to the conduction band, but weakly bound to the donor sites. These states are hydrogen-like, but with an expanded Bohr radius \( a_B \approx 15 - 30 \ \text{Å} \) and reduced binding energy \( E_b \approx 10 - 50 \ \text{meV} \).

The electron of the \(^{31}\text{P}\)-atom has spin 1/2 and silicon \(^{28}\text{Si}\) has spin 0. The nucleus-electron system \( \text{Si:P} \) with an external magnetic field \( B \) along the \( z \)-axis is thus:

\[
H_{en} = \mu_B B \sigma_z^e - g_n \mu_n B \sigma_z^n + A \sigma^n \cdot \sigma^e
\]

where \( \sigma \) are Pauli spin matrices for the electron (index \( e \)) and spin (Index \( n \)), \( \mu_B \) and \( \mu_n \) are the Bohr and nuclear magneton, \( g_n \) the nuclear g-factor (for the electrons a g-factor of 2 is approximated), and \( A \) is the contact hyperfine interaction energy

\[
A = \frac{8}{3} \pi \mu_B g_n |\Psi(0)|^2
\]

with the probability density of the electron wave function evaluated at the nucleus \( |\Psi(0)|^2 \).

The following figure shows the spin energy levels of the P-atom as a function of the magnetic field. The singlet and the triplet states are easily identified.
Some advantages of the P:Si system are:

- Low spontaneous transition rates between the energy levels:
  At temperatures of 1 K the transition time is on the order of 1 hour. Thus, the states can be regarded as meta-stable.

- Narrow linewidth of RF-induced transitions:
  In isotopically purified $^{28}$Si (silicon has an isotope $^{29}$Si with $I = 1/2$) the linewidth is $\leq 1 \text{MHz}$.

- Low dephasing time $T_2$.
  $T_2$ is on the order of 0.5 m sec in isotopically purified Si, probably due to dipolar interactions with other electrons. Decoupling algorithms can be used to further increase $T_2$.

### 4.5.1 Single qubit gates

A qubit is represented by the nuclear spin of the P-atom. Initialization of the qubits is achieved by cooling the sample to very low temperatures in a strong magnetic field. However, active cooling techniques have been suggested, which use controlled coupling to a spin polarized reservoir.

In order to drive single spin rotations a constant RF-field with fixed frequency is applied. Transition are switched on and off resonance with the help of so-called A-gates. The inset of the following picture illustrates the A-gate:
The electron wavefunction can be distorted if a positive voltage is applied to the A-gate as shown in the figure. This changes the electron density at the site of the P-atom $|\Psi(0)|^2$ and thus $A$. Therefore, the resonance frequency for transitions between the spin states can be controlled, and controlled single qubit rotations are possible.
4.5.2 Two-qubit gates

Two qubit gates are realized by a controlled interaction between two qubits, or the two nuclear spins in this case. If two nuclei share a common electron "cloud" then there is an effective interaction between the nuclei mediated by the electrons. This is the so-called J-coupling which is introduced in NMR. Kane proposed to implement so-called J-gates to control the J-coupling between neighboring P-atoms. The following figure shows the schematics:

![Figure 88: Principle of a J-gate. [from Kane, Fortschr. Phys. 48, 1023 (2000)]](image)

As described above a positive voltage on an A-gate spreads out the electronic wavefunction. If two P donors are very close to each other the electron wave function of both atoms can overlap. In order to establish an independent control of single and two qubit interaction, a third J-gate is placed between two A-gates. This J-gate now controls the amount of overlap and thus the amount of J-coupling.

The strength of the J-coupling energy as a function of the distance between neighboring nuclei can be approximated as

\[ J(r) \sim E_b \left( \frac{r}{a_B} \right)^{5/2} \exp \left( -\frac{2r}{a_B} \right) \]

This formula can be derived from a model of two well-separated H-atoms. Figure 89 illustrates the behaviour:
The J-coupling energy can be very large (up to \(10^{11} \text{ Hz}\)) and can be switched on and off. The controlled two-qubit interaction together with single qubit rotations allows arbitrary quantum computation!

The speed of a silicon quantum computer is limited by single qubit operations which can be performed at a rate of \(\approx 10 \sim 100 \text{ kHz}\).

### 4.5.3 Single spin measurement

A very difficult task in the proposed structure is the required read-out of the state of a single spin. A solution could be to transform the problem into the read-out of charge with the help of a single-electron transistor (SET). As discussed in the previous chapter an SET is a very sensitive electrometer capable of detecting single charges.

The proposed strategy is to read-out a two-electron system \(|\uparrow\uparrow\rangle\), where one electron is in a known spin state. Techniques to prepare electrons in a well defined spin state are known and first experiments have been performed.
A two-electron system can be in one of the following states:

\[
\text{singlet} : \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \\
\text{triplet} : |\uparrow\uparrow\rangle \text{ or } |\downarrow\downarrow\rangle \text{ or } \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) 
\]

The scattering from the state \(\frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)\) to the state \(\frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)\) is much faster than the scattering between \(|\uparrow\uparrow\rangle\) and \(|\downarrow\downarrow\rangle\) since in the latter a spin flip is required. Thus after some time it is sufficient to discriminate a singlet from a triplet state and to detect the unknown spin. Doubly charged donor atoms such as column VI impurities in silicon (e.g. Te) can bind two-electron states. The singlet state is much more strongly bound than the triplet state. This can be used to build a singlet/triplet detector with a column VI donor in silicon and an SET as shown in the following figure:

![Figure 90: Schematics of a singlet/triplet detector made from a column VI donor in Si. [from Kane, Fortschr. Phys. 48, 1023 (2000)]](image)

If no external electric field is applied then both singlet and triplet states are bound to the donor site. However, if the electric field strength is increased
then first the triplet state is ionized. The ionized electron will be located close to the Si/barrier interface, where it is easily detected by the SET-electrometer. A main drawback of this proposal is that the spin state has to be transferred coherently from the qubit donor site to the read-out donor site. This could be established by subsequent swapping between neighboring qubits, but is still a non-trivial task.

Silicon quantum computation is still in its very infancy. However, the individual tasks (control of single donor sites, manipulation of single donors, read-out of single spin) are within the reach of present day technology. Additionally, these tasks are highly important for the fabrication of smaller and smaller conventional computers, which will strongly drive attempts to overcome the problems.
4.6 Quantum Dots as Qubits

In quantum dots electrons are confined in all three directions. If the confining structure is small enough the discrete energies of single electron states can be resolved. Due to the analogy with single electrons trapped in the Coulomb potential of a single nucleus, quantum dots are sometimes called artificial atoms. The attractive aspect - compared to single atoms - is that the confining potential can be manipulated or even designed at will. There are mainly two different kinds of quantum dots:

- Electrically defined quantum dots
  In these dots tiny electrodes confine the electronic motion of a two-dimensional electronic system. The two-dimensional system is either a two-dimensional degenerate electron gas (2DEG) or a thin layer of semiconductor material sandwiched between semiconductors of a higher band-gap (quantum well).

- Three dimensional semiconductor dots
  Here, electrons are confined in a nanoscopic amount of semiconductor material surrounded by some matrix material (solution, polymer, glass or a higher-bandgap semiconductor)

The following picture shows the band diagram of an interface of GaAs and Al$_x$Ga$_{1-x}$As.

![](image.png)

Figure 91: A 2DEG created in a hetero-junction of n-doped AlGaAs and GaAs. The electrons are confined in a triangular shaped potential.

Under certain conditions a thin layer of electrons is formed close to the interface. In this layer the electrons form a quasi two-dimensional system, a
so-called degenerate quantum gas. Additional negatively charged electrodes on
top of such a gas confine the electrodes laterally. Figure 92 shows an already
rather complex design where 5 quantum dots are defined close to each other.

Figure 92: Five coupled quantum dots produced by electric contacts on top of
a 2DEG [from http://www.nano.physik.uni-muenchen.de/Blick/dots.html]

4.6.1 Fabrication of quantum dots:

Electrically defined quantum dots are made by writing metal electrodes on two-
dimensional systems using electron beam lithography, metal evaporation and
lift-off techniques.
The following picture shows three possible ways to fabricate three dimensional dots:

![Diagram](image_url)

Figure 93: Different approaches to fabricate quantum dots (a) chemical synthesis, (b) artificial patterning of thin film structures, (c) self-organized growth [from D. Bimberg et al., Wiley 1999]

1. Semiconductor quantum dots or (nanocrystallites) can be grown chemically in a glass or various solutions. The typical diameter of these dots is in the order of a few nanometers. The best studied systems so far are II-VI quantum dots (CdSe and CdS).

2. A post can be etched out of a material that contains a quantum well. The disadvantage here is that the confined electrons can interact strongly with surface defects.

3. Under a certain growth condition of a material on a substrate with a different lattice constant (Stranski-Krastanov mode in Molecular Beam epitaxy, MBE, or Metal-organic vapor phase epitaxy, MOVPE) little "droplets"
of semiconductor material form automatically. These quantum dots are sometimes referred to as self-organized quantum dots. Presently, these dots provide the highest quality with respect to optical efficiency and coherence times.

The following picture shows a few InAs quantum dots on a GaAs surface:

![Atomic force microscope picture of single InAs quantum dots on a GaAs surface](http://w3.rz-berlin.mpg.de/pc/ElecSpec/)

![Scanning electron microscope picture of a single InAs quantum dot](http://w3.rz-berlin.mpg.de/pc/ElecSpec/)

Figure 94: Atomic force microscope picture of single InAs quantum dots on a GaAs surface [from http://w3.rz-berlin.mpg.de/pc/ElecSpec/]

Figure 95: Scanning electron microscope picture of a single InAs quantum dot [from http://w3.rz-berlin.mpg.de/pc/ElecSpec/]
4.6.2 Single qubit gates:

The following design shows a proposal by Daniel Loss’ group (G. Burkhard et al., Fortschr. Phys. 48, 965 (2000)) for qubits made from coupled quantum dots.

Figure 96: Schematics of an all-electrically controlled quantum dot array [from G. Burkhard et al. Fortschr. Phys. 48, 965 (2000)]

Quantum dots are defined by gate electrodes on top of a quantum well. If a magnetic field is applied perpendicular to the plane of the quantum well a Zeeman splitting between spin-up and spin down-electrons occurs. Transitions between these two qubit levels can be obtained by applying an RF-field. This electron spin resonance (ESR) is in perfect analogy to nuclear magnetic resonance (NMR). Similar as in the case of single donors the ESR resonance frequency can be changed by moving the confined electrons in the direction of a high-g magnetic layer below the quantum well by electrical gating. Thus the electrons can be switched on and off resonance at will. If a constant RF-field with fixed frequency is applied this leads to the realization of arbitrary single qubit rotation.

Another way of coupling dots is used with self-organized dots. The self-organization is driven by the lattice mismatch of the substrate and quantum dot material, respectively. If an additional layer of quantum dots is grown on top of a first layer, then the localized strain caused by the buried quantum dots in the first layer acts as a ”seed” for self-organized growth in the second layer. Then, quantum dots predominantly grow as stacks on top of each other.
Figure 97: Transmission electron microscopy (TEM) pictures of a single (a) and three (b) layers of quantum dots. The dots tend to be stacked on top of each other. [from Zundel et al. Appl. Phys. Lett. 71, 2972 (1997)]

The coupling (J-energy) between stacked dots of different size can be switched on and off by applying an in-plane electric field as illustrated in the following figure.

Figure 98: Controlling the coupling (J-energy) of two stacked dots of different size with an in-plane electric field. [from Zundel et al. Appl. Phys. Lett. 71, 2972 (1997)]
4.6.3 Two-qubit gates:
As already illustrated in the previous subsection two qubit gates are implemented by lowering the potential barrier between adjacent dots. Then tunneling occurs similar as in the Josephson charge qubit. An effective two dot molecule is formed. After a well defined time the tunneling barrier can be raised again. Without giving details it is obvious that this procedure allows arbitrary two-qubit operation. In summary, arbitrary quantum gates can be realized this way. The limit of switching speed is given by single qubit rotations, which can be performed within a few 10 psec.

4.6.4 Initialization and read-out
Initialization of n qubits can be realized thermally if the relation
\[ g\mu_B B \gg kT \]
is fulfilled. In contrast to nuclear magnetic resonance where only \( B \) (liquid phase) or \( B \) and \( T \) (solid phase) can be changed, here manipulation of \( g \) is possible as well.

More advanced methods have been proposed where spin polarization is achieved by coupling of the qubits to a spin polarized reservoir.

Similar as in the case of silicon quantum computing the problem of single spin detection could be circumvented by translating the spin into a charge information. Additionally, possible elements like spin selective barriers have been discussed which would be similar to polarizers in optical devices.
4.6.5 Optical manipulation of quantum dots

Semiconductor quantum dots offer the possibility of optical manipulation. The following picture shows the fluorescence intensity of a sample of single quantum dots. At low density (single quantum dot level) sharp lines could be identified, which correspond to optical transitions between the discrete energy levels of electrons and holes in the quantum dots. The observation of this discrete spectrum also motivated the notation "artificial atoms".

Figure 99: Photoluminescence (PL) spectrum of InAs quantum dots on a GaAs substrate. (a) shows an ensemble of many dots whereas in (b) discrete lines from single dots are resolved. (d) is the same as (b), but with a lower excitation intensity. [from D. Bimberg et al. Wiley 1999]

With the help of narrow-band laser it is thus possible to excited a single electron hole pair. A qubit in this system could also be represented by

\[
|0\rangle = |0\rangle_{\text{electron}} |0\rangle_{\text{hole}} \\
|1\rangle = |1\rangle_{\text{electron}} |1\rangle_{\text{hole}}
\]

Single qubit rotations are realized by applying optical pulses of a certain pulse area. One obstacle of using optical excitations as qubits is the relatively short coherence time which is given by the spontaneous decay time of about 1 nsec.
Figure 100: Photoluminescence of a single quantum dot as a function of the excitation power [from V. Zwiller et al., Appl. Phys. Lett. 78, 2476 (2001)]

which has to be compared to the spin decoherence time (e.g. of a single electron in a quantum dot) which is believed to exceed 100 $n$sec.

Nevertheless, since the polarization of the light and the spin state of the electron-hole pair are correlated, optical manipulation is attractive for the following reasons:

- The spin state of the electron can be controlled by the polarization of the laser light which is used for excitation.
- An optical single spin measurement is possible by measuring the polarization of the fluorescence light or by detecting the Faraday rotation of an optical probe pulse.
- Quantum information, which can be more easily manipulated in an electronic system, may be transferred into an optical system (photons). Photons are ideal transmitters of quantum information. Possible realization of such quantum electro-optical interfaces are the topic of present research.
• Cavity quantum electrodynamics (QED) can be combined with quantum information processing in solid state systems.

Finally, the following illustrates a proposal (A. Imamoglu et al., Phys. Rev. Lett. 83, 4204 (1999)) where optical manipulation of the spin of single electrons in quantum dots with the help of cavity-QED effects is suggested. The proposed structure consists of a microdisc resonator which contains several quantum dots (see schematics below). These structures have been realized successfully (see figure below).

![Schematics of a microdisk resonator containing a few quantum dots.](image)

The proposal relies on the possibility to excite optical Raman transitions which induce a spin flip of a single electron in a quantum dot. The dots are embedded in a microdisk and can be excited individually with the help of optical fibers. Qubits are represented by the electron spin eigenstates along the $x$-axis which is defined as the quantization axis by a magnetic filed $B_x$. Single qubit gates are Raman rotations when the two-photon resonance condition is fulfilled for a single quantum dot. Two qubit gates are established if the Raman resonance condition is fulfilled between two quantum dots which share a common mode of the microdisk resonator.
Since the optically excited states are excited only virtually, decoherence is much reduced in this system. An effective spin-spin interaction between two quantum dots can be realized. The following shows a schematics of the transitions which are involved, where $\omega_L$ and $\omega_{cav}$ denote the two laser and cavity fields, respectively.
More recent approaches utilize so-called photonic crystal membrane structures with embedded quantum dots [A. Badolato et al., Science 20, 1158 (2005)]. Photonic crystals are dielectrics with a periodic modulation of the index of refraction. Energy bands occur where light with a frequency within the band cannot propagate anymore. A realization of a photonic crystal is a free standing membrane with periodic holes. Missing holes introduce defects of the periodicity. Close to these defects light can be tightly confined as in a resonator. As shown in Fig. 104 it is possible to couple only a single quantum dot to such a resonator. This is the solid-state analogy of a single trapped atom in an optical cavity.
4.7 Summary: Solid State Implementation

- **Qubit representation:** (a) Josephson qubits: Charge or flux. (b) Silicon QC and quantum dots: Electron spin.

- **Unitary evolution:** (a) Josephson qubits: Arbitrary single qubit operation is obtained by switching on the $B_x$ and $B_z$ part of the single-qubit Hamiltonian. This is achieved by changing a gate voltage or controlling an external magnetic flux. Controllable two-qubit operation between arbitrary two qubits is established by coupling qubits with a common $LC$ circuit. (b) Silicon QC and quantum dots: Single qubit rotations are induced by RF-spin resonance pulses. Usually, the RF-frequency is kept constant and the resonance frequency is switched. Two-qubit gates are implemented by coupling the two spins. In order to achieve this either a barrier between the separated qubits is lowered or the electronic wave functions are spread out and overlap for a certain time.

- **Initial state preparation:** (a) Josephson qubits: Initial state preparation at low temperature can be realized by setting appropriate gate voltages or applying short gate voltage pulses. (b) Silicon QC and quantum dots: Cooling in a strong magnetic field or active spin polarization by coupling the qubits to a spin polarized reservoir.

- **Readout:** (a) Josephson qubits: Measurement of a current through a
probe gate or determining the charge with the help of a single-electron transistor. (b) Silicon QC and quantum dots: Detection of singlet/triplet state and charge detection with a single-electron transistor. Spin selective barriers.

- **Drawbacks:** (a) Josephson qubits: Low temperature operation (dilution refrigerators) is required to reduce the thermal energy below single-electron charging energy or "single-flux" Josephson energy. Charged particles are very sensitive to background charges. The flux is sensitive to stray magnetic fields. (b) Silicon QC and quantum dots: Low temperature operation. Fast decoherence rates require very fast switching. Single spin detection is very difficult.